Verification of Galois Field Multipliers

A Gröbner Bases + Nullstellensatz Approach

Priyank Kalla

Department of Electrical and Computer Engineering

University of Utah, Salt Lake City

Introduction

- Elliptic Curve Cryptography (ECC) requires multiplication and exponentiation
- Operations performed over finite (Galois) fields, $GF(2^k)$
- Hardware VLSI implementations of multipliers and their optimization
- Mastrovito Multiplication, Galois Field Decomposition $GF(2^k) \equiv GF((2^m)^n)$, Montgomery multiplication, etc.
- So we have a Verification Problem: Spec $\equiv$ implementation?
- Problem is hard: $m = 256$ bits, can be more
- BDD/SAT/SMT are infeasible, so we explore Computer Algebra!

Galois Fields

A **Galois Field** is a set F_q, satisfying all the following properties:

- *Abelian Group:* w.r.t. addition "+", and 0 element
- *Commutative ring with unity:* $(+, \times, 1, 0)$
- Associativity, Commutativity, Distributivity
- *Inverse:* $\forall a \in F_q - \{0\},\ \exists a^{-1} \in F_q$ such that $a \cdot a^{-1} = 1$.
- $q = p^m$, where p is prime. In our case, $p = 2$.
- Multiplicative cyclic group structure: $a^q = a$.

$(\mathbb{Z} \pmod p)$, where $p =$ prime is a field.

Extension Fields

If D is a Euclidean domain, and p is a prime in D, then $D \pmod{p}$ is a field.

- $(\mathbb{Z} \pmod{p})$, where $p =$ prime is a field. We call it $\mathbb{Z}_p \equiv F_p \equiv GF(p)$.
- $D = \mathbb{R}[x], p = x^2 + 1$, we have $\mathbb{R}[x] \pmod{x^2+1} = \mathbb{C}$, the field of complex numbers.
- $D = \mathbb{Z}_p$ and we take an irreducible polynomial $f(x)$ of degree m, irreducible in $\mathbb{Z}_p$, then $\mathbb{Z}_p \pmod{f(x)} = F_{p^m}$ or $GF(p^m)$.
- Consider $GF(p^m)$ as an m-dimensional vector space over $GF(p)$.
- Example: $GF(2) \pmod{x^3 + x + 1}$ is $GF(2^3)$.
- Galois Fields are unique up to the labeling of elements.

Field Elements

Consider: $GF(2^3)$ with irreducible polynomial $p(x) = x^3 + x + 1$. Let $A \in F_2[x]$ and compute $A \pmod{p(x)} = a_2x^2 + a_1x + a_0$, where $a_2, a_1, a_0 \in \{0, 1\}$. Let $p(\alpha) = 0$:

- $\langle a_2, a_1, a_0 \rangle = \langle 0, 0, 0 \rangle = 0$
- $\langle a_2, a_1, a_0 \rangle = \langle 0, 0, 1 \rangle = 1$
- $\langle a_2, a_1, a_0 \rangle = \langle 0, 1, 0 \rangle = \alpha$
- $\langle a_2, a_1, a_0 \rangle = \langle 0, 1, 1 \rangle = \alpha + 1$
- $\langle a_2, a_1, a_0 \rangle = \langle 1, 0, 0 \rangle = \alpha^2$
- $\langle a_2, a_1, a_0 \rangle = \langle 1, 0, 1 \rangle = \alpha^2 + 1$
- $\langle a_2, a_1, a_0 \rangle = \langle 1, 1, 0 \rangle = \alpha^2 + \alpha$
- $\langle a_2, a_1, a_0 \rangle = \langle 1, 1, 1 \rangle = \alpha^2 + \alpha + 1$

Field Elements

Multiply two elements: $(\alpha^2+1)(\alpha^2+\alpha)$ modulo $p(x)=\alpha^3+\alpha+1$:

$$
\begin{aligned}
& (\alpha^2+1)(\alpha^2+\alpha) \\
= \quad & \alpha^4+\alpha^3+\alpha^2+\alpha \\
= \quad & \alpha(\alpha^3)+\alpha^3+\alpha^2+\alpha \\
= \quad & \alpha(\alpha+1)+(\alpha+1)+\alpha^2+\alpha \\
= \quad & \alpha^2+\alpha+\alpha+1+\alpha^2+\alpha \\
= \quad & \alpha+1
\end{aligned}
$$

Mastrovito Multiplier in GF(2^4)

Input:

$$A(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3$$
$$B(x) = b_0 + b_1 \cdot x + b_2 \cdot x^2 + b_3 \cdot x^3$$

Irreducible Polynomial:

$$P(x) = x^4 + x^3 + 1$$

Result:

$$A(x) \times B(x) \pmod{P(x)}$$

Mastrovito Multiplier over GF(2^4)

			a_3	a_2	a_1	a_0
$\times$			b_3	b_2	b_1	b_0
			$a_3 \cdot b_0$	$a_2 \cdot b_0$	$a_1 \cdot b_0$	$a_0 \cdot b_0$
		$a_3 \cdot b_1$	$a_2 \cdot b_1$	$a_1 \cdot b_1$	$a_0 \cdot b_1$	
	$a_3 \cdot b_2$	$a_2 \cdot b_2$	$a_1 \cdot b_2$	$a_0 \cdot b_2$		
$a_3 \cdot b_3$	$a_2 \cdot b_3$	$a_1 \cdot b_3$	$a_0 \cdot b_3$			
s_6	s_5	s_4	s_3	s_2	s_1	s_0

In polynomial expression:

$$s_0 + s_1 \cdot x + s_2 \cdot x^2 + s_3 \cdot x^3 + s_4 \cdot x^4 + s_5 \cdot x^5 + s_6 \cdot x^6$$

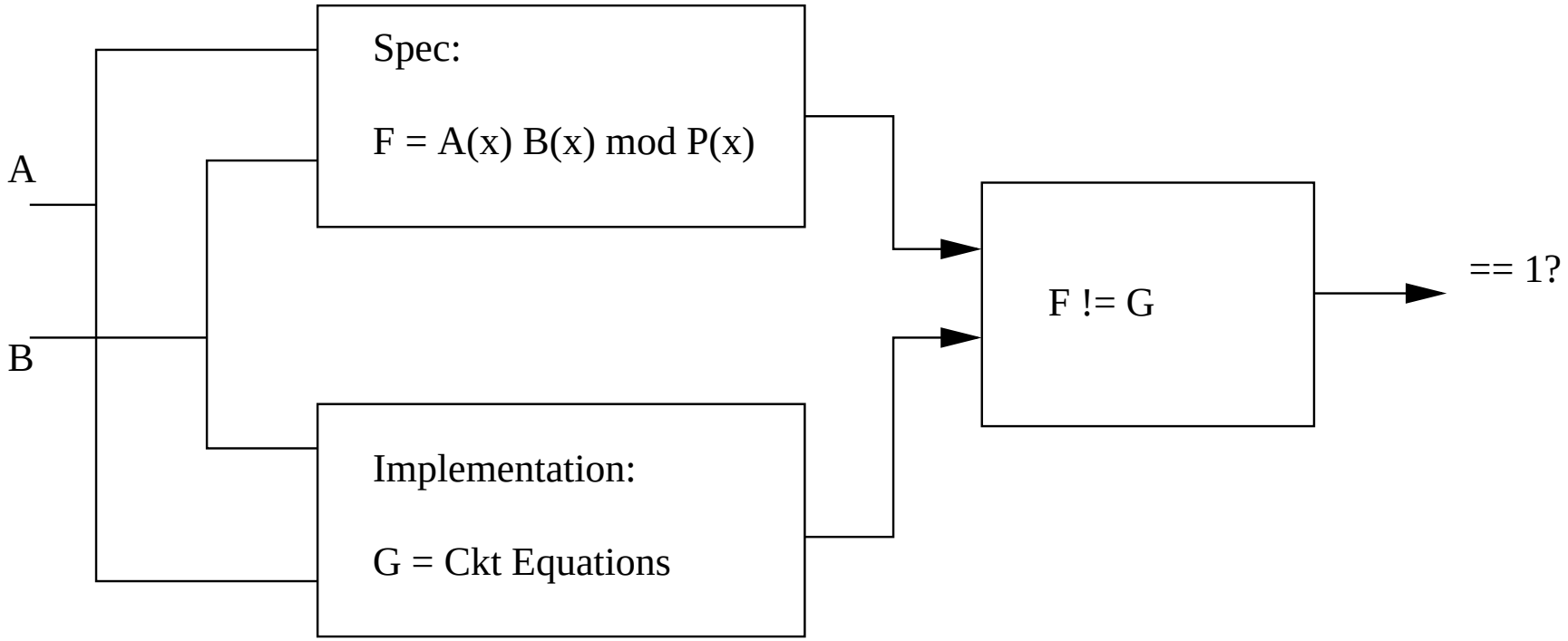
Mastrovito Multiplier over GF(2^4)

s_6	s_5	s_4	s_3	s_2	s_1	s_0		
			s_4	0	0	s_4	$\Leftarrow$	$s_4 \cdot x^4 \pmod{P(x)}$
			s_5	0	s_5	s_5	$\Leftarrow$	$s_5 \cdot x^5 \pmod{P(x)}$
		$+$	s_6	s_6	s_6	s_6	$\Leftarrow$	$s_6 \cdot x^6 \pmod{P(x)}$
			r_3	r_2	r_1	r_0		

Verification Problem is to check if:

$$A(x) \times B(x) \equiv r_3x^3 + r_2x^2 + r_1x + r_0 \pmod{P(x)} \text{ ?}$$

Verification Setup



To model $F \neq G$ as a polynomial over $GF(2^m)$:

$t(F - G) = 1$, where t is a new variable in $GF(2^m)$.

- When $F = G, (F - G) = 0$, so $t \cdot 0 = 1$ has no solutions.
- When $F \neq G, (F - G) = t^{-1} \neq 0, t \cdot t^{-1} = 1$.

SAT/SMT or Computer Algebra?

Our Problem: Given a set of polynomial equations, do they have a solution?

- If solutions exists: BUG! Otherwise, equivalence is proven.
- SAT/SMT solvers: Search for a solution, solve systems of equations.
- In ECC, $GF(2^m), m = 256$ word-length of operands.
- SAT/SMT is infeasible.
- Our experience: SAT fairs better than SMT for this problem. Can solve $m = 16$ but not $m = 32$.

Use symbolic computer algebra and algebraic geometry to reason whether a given system of polynomials has a solution or not?

Terminology

Let $k = F_q = GF(q = 2^m)$:

- $k[x_1, \ldots, x_n]$ denotes the ring of all polynomials with coefficients in k.
- Given a set of polynomial equations:
  - $f_1, f_2, \ldots, f_s \in k[x_1, \ldots, x_n]$
  - Find solutions to $f_1 = f_2 = \cdots = f_s = 0$
- **Variety:** Set of ALL solutions to a given system of polynomial equations: $V(f_1, \ldots, f_s)$
- In general, variety can be infinite, or finite non-empty set, or empty.
- Is the variety empty, i.e. is $V(f_1, \ldots, f_s) = \emptyset$?

Variety

Given a field k and $n \in Z^+$, we define n-dimensional affine space over k to be the set:

$k^n = \{(a_1, \ldots, a_n) : a_1, \ldots, a_n \in k\}$

In case $k^2 = R^2$, we get our affine plane.

Let k be a field, and $f_1, \ldots, f_s$ are polynomials in $k[x_1, \ldots, x_n]$.

Then we set:

$\mathbf{V}(f_1, \ldots, f_s) = \{(a_1, \ldots, a_n) \in k^n : f_i(a_1, \ldots, a_n) = 0, 1 \leq i \leq s\}$

We call $\mathbf{V}(f_1, \ldots, f_s)$ the affine variety defined by the polynomials.

Variety = Set of all solutions to a given set of polynomial equations!

Variety

Consider the points $\{(a_1, \ldots, a_n) : a_1, \ldots, a_n \in k\}$ in F_q^n:

- Any single point is a variety of some polynomial system
- $(a_1, \ldots, a_n)$ is a variety of $x_1 - a_1,\ x_2 - a_2, \ldots,\ x_n - a_n$
- **Finite unions and intersections** of varieties is a variety
  - Let $V = \mathbf{V}(f_1, \ldots, f_s)$ and $W = \mathbf{V}(g_1, \ldots, g_t)$
  - $V \cap W = \mathbf{V}(f_1, \ldots, f_s, g_1, \ldots, g_t)$
  - $V \cup W = \mathbf{V}(f_i g_j : 1 \leq i \leq s, 1 \leq j \leq t)$
- F_q^n is a finite n-dimensional affine space, so it is also a variety

Variety depends not just on the given system of polynomial equations, but rather on the **ideal** generated by the polynomials.

Ideals

Definition 1 *A subset* $I \subset R = k[x_1, \ldots, x_n]$ *is an* **ideal** *if:*

- $0 \in I$
- *If* $f, g \in I$*, then* $f + g \in I$
- *If* $f \in I$ *and* $h \in R$ *then* $f \cdot h \in I$

Definition 2 *Let* $f_1, f_2, \ldots, f_s \in k[x_1, \ldots, x_n]$*. Let*

$$\langle f_1, f_2 \ldots, f_s \rangle = \{\sum_{i=1}^{s} f_i h_i : h_1, \ldots h_s \in k[x_1, \ldots, x_n]\} \tag{1}$$

$$\langle f_1, f_2 \ldots, f_s \rangle = f_1 h_1 + f_2 h_2 + \cdots + f_s h_s \tag{2}$$

$I = \langle f_1, f_2 \ldots, f_s \rangle$ *is an ideal generated by* $f_1, \ldots, f_s$ *and the polynomials are called the generators.*

Variety depends on the Ideal

Let $V(f_1, \ldots, f_s) = (a_1, \ldots, a_n) \in F_q^n$. Consider the ideal $\langle f_1, f_2 \ldots, f_s \rangle$ evaluated at $(a_1, \ldots, a_n)$:

$$\begin{aligned}
& \langle f_1, f_2 \ldots, f_s \rangle \\
= \quad & \{\sum_{i=1}^{s} f_i h_i : h_1, \ldots h_s \in k[x_1, \ldots, x_n]\} \\
= \quad & f_1 h_1 + f_2 h_2 + \cdots + f_s h_s \\
= \quad & f_1(a_1, \ldots, a_n) h_1 + \cdots + f_s(a_1, \ldots, a_n) h_1 \\
= \quad & 0
\end{aligned}$$

Any polynomial in the ideal, vanishes at its variety!

Ideals and their Generators

Consider ideal $I \subset F_q[x_1, \ldots, x_n]$:

- Every ideal in $F_q[x_1, \ldots, x_n]$ is finitely generated
- Also true for $\mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}, \mathbb{Z}_p$: Noetherian rings
- An ideal I has many different bases: $I = \langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle$
  - Variety depends on the ideal:
  - $V(I) = V(f_1, \ldots, f_s) = V(g_1, \ldots, g_t)$
  - Remember Gaussian elimination?
- Some generators (basis) are “better” than others: nice properties, easy to solve, easy to reason whether variety is finite, infinite or empty
- Gröbner basis is one such “nice basis”, which helps to reason about varieties!

Ideal-Variety Examples

- $I_1 = \langle f_1, f_2 \rangle \subset Q[x, y]$
- $f_1 = x^2 - 4; \quad f_2 = y^2 - 1$
- $I_2 = \langle g_1, g_2 \rangle \subset Q[x, y]$
- $g_1 = 2x^2 + 3y^2 - 11; \quad g_2 = x^2 - y^2 - 3;$
- $g_1 = 2f_1 + 3f_2; \; g_2 = f_1 - f_2; \implies g_1, g_2 \in I_1$, so $I_2 \subseteq I_1$.
- Similarly, show that $f_1, f_2 \subseteq I_2$
- If $I_1 \subset I_2$, and $I_2 \subset I_1$ then $I_1 = I_2$
- Note $V(I_1) = V(I_2) = \{(\pm 2, \pm 1)\}$

Ideal Membership Testing

Definition 3 *Let* $f, f_1, f_2, \ldots, f_s \in k[x_1, \ldots, x_n]$. *Let* $I = \langle f_1, f_2 \ldots, f_s \rangle$. *If we have:*

$$f = \sum_{i=1}^{s} f_i h_i : h_1, \ldots h_s \in k[x_1, \ldots, x_n] \tag{3}$$

then we say that $f \in I$ *and we can then write* $f = f_1 h_1 + \cdots + f_s h_s$. *IOW, we ask: Is* f *a member of the ideal generated by* $f_1, \ldots, f_s$*? This is ideal membership testing!*

- If $f = f_1 h_1 + \cdots + f_s h_s$ then $f \in I$
- Intuitively, this is like dividing f by $f_1, \ldots, f_s$, getting quotients $h_1, \ldots h_s$ and zero remainder.
- But what if we cannot find such h_i and zero remainder via division? Does that mean $f \notin I$? Not really, as the following example shows...

Ideal Membership Testing

- Example: $f = x;\ f_1 = x^2; f_2 = x^2 - x;\ \ f = f_1 - f_2$, so we have $f \in I = \langle f_1, f_2 \rangle$. However, both f_1, f_2 cannot divide f, as f has smaller degree than both f_1, f_2.
- The above case depicts one of the problems with the given "generators" f_1, f_2 of I – i.e. they are not a "good" generating set. Gröbner basis of $I =< f_1, f_2 >=< x >$ and x generates the same ideal and also allows to decide whether or not f is in I?
- For univariate polynomial rings: $I =< f_1, \ldots, f_s >=$ GCD$(f_1, \ldots, f_s)$. Then $f \in I \iff f = h \cdot$ GCD$(f_1, \ldots, f_s)$.
- For multivariate polynomials, this is much harder!

Monomial Ordering

In multivariate polynomials, we need to order the monomial terms in some specific order $x^\alpha = (x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_n^{\alpha_n})$:

- Total order: One and only one of the following should be true: $x^\alpha > x^\beta$ or $x^\alpha = x^\beta$ or $x^\alpha < x^\beta$.
- $1 < x^\alpha, \quad \forall x^\alpha \quad (x^\alpha \neq 1)$
- $x^\alpha < x^\beta \implies x^\alpha \cdot x^\gamma < x^\beta \cdot x^\gamma$.

Definition 4 **Lexicographic order:** *Let* $x_1 > x_2 > \cdots > x_n$ *lexicographically. Also let* $\alpha = (\alpha_1, \ldots, \alpha_n)$; $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$. *Then we have:*

$$x^\alpha < x^\beta \iff \begin{cases} \textit{Starting from the left, the first co-ordinates of } \alpha_i, \beta_i \\ \textit{that are different satisfy } \alpha_i < \beta_i \end{cases} \tag{4}$$

Ordering Examples

Let $f = 2x^2yz + 3xy^3 - 2x^3$

- LEX $x > y > z$: $f = -2x^3 + 2x^2yz + 3xy^3$
- DEGLEX $x > y > z$: $f = 2x^2yz + 3xy^3 - 2x^3$
- DEGREVLEX $x > y > z$: $f = 3xy^3 + 2x^2yz - 2x^3$

In singular: we declare ordering as:

```
ring r = 0, (x, y, z), lp; //LEX x > y > z
ring r = 0, (x, z, y), Dp; //DEGLEX x > z > y
ring r = 0, (y, z, x), dp; //DEGREVLEX y > z > x
```

Multivariate Division

$$
\begin{array}{l|l}
 & -3x + y^2 \\
\hline
x + 2y + 1 & -3x^2 + y^2 x + 4xy \\
 & -3x^2 + 3x - 6xy \\
\cline{2-2}
 & y^2 x + 10xy - 3x \\
 & \ldots\ldots \\
 & \ldots\ldots \\
\cline{2-2}
 & 2y^3 - 19y^2 - 4y + 3
\end{array}
$$

- To divide f by g, we denote it as $f \xrightarrow{g} h$
- where $h = f - \frac{\mathsf{lt}(f)}{\mathsf{lt}(g)} g$.

Multivariate Division

Definition 5 *Let* $f, f_1, \ldots, f_s, h \in k[x_1, \ldots, x_n], f_i \quad \neq 0; \quad F = \{f_1, \ldots, f_s\}$. *Then* f *reduces to* h *modulo* F*:*

$$f \xrightarrow{F}_{+} h$$

if and only if there exists a sequence of indices $i_1, i_2, \ldots, i_t \in \{1, \ldots, s\}$ *and a sequence of polynomials* $h_1, \ldots, h_{t-1} \in k[x_1, \ldots, x_n]$ *such that*

$$f \xrightarrow{f_{i_1}} h_1 \xrightarrow{f_{i_2}} h_2 \xrightarrow{f_{i_3}} \cdots \xrightarrow{f_{i_{t-1}}} h_{t-1} \xrightarrow{f_{i_t}} h$$

Definition 6 *If* $f \xrightarrow{F}_{+} r$*, then no term in* r *is divisible by LT*(f_i), $\forall f_i \in F$. *Then* r *is reduced w.r.t.* F *as is called the remainder.*

Division Algorithm

Definition 7 *Let* $f, f_1, \ldots, f_s, r \in k[x_1, \ldots, x_n], f_i \neq 0; \quad F = \{f_1, \ldots, f_s\}$. *Then* f *reduces to* r *modulo* F*:*

$$f \xrightarrow{F}_{+} r$$

then

$$f = u_1 f_1 + \cdots + u_s f_s + r$$

then we have

- r *is reduced w.r.t.* F
- $u_1, \ldots u_s \in k[x_1, \ldots, x_n]$
- $LP(f) = MAX(LP(f_1)LP(u_1), \ldots LP(f_s)LP(u_s), r)$

Subtleties of the Division Process

Suppose we have: $f \xrightarrow{F}_{+} r$. If $r = 0$ then $f \in I =< F >$. But, if $r \neq 0$ then we cannot decided ideal membership unequivocally. That's why we need Groebner Bases.

Let $f = xy^2 - x;\ f_1 = xy + 1;\ f_2 = y^2 - 1$. Note that $f \in I = \langle f_1, f_2 \rangle$ as $f = xf_2$.

$$f \xrightarrow{f_1, f_2}: f = y \cdot f_1 + 0 \cdot f_2 + (-x - y)$$

$$f \xrightarrow{f_2, f_1}: f = x \cdot f_2 + 0 \cdot f_1 + 0$$

Moral of the Story If we change the order in which we divide by $f_1, \ldots, f_s$, we obtain different quotients and different remainders.

Motivating Gröbner Bases

Let $F = \{f_1, \ldots, f_s\}$; $I = \langle f_1, \ldots, f_s \rangle$ and let $f \in I$. Then we should be able to represent $f = u_1 f_1 + \cdots + u_s f_s + r$ where $r = 0$. If we were to divide f by $F = \{f_1, \ldots, f_s\}$, then we will obtain an intermediate remainder (say, h) after every one-step reduction. The leading term of every such remainder (LT(h)) should be divisible by the leading term of at least one of the polynomials in F. Only then we will have $r = 0$.

Definition 8 *Let* $F = \{f_1, \ldots, f_s\}$; $G = \{g_1, \ldots, g_t\}$; $I = \langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle$. *Then* G *is a* **Gröbner Basis** *of* I

$$\Longleftrightarrow$$

$$\forall f \in I \;\; (f \neq 0), \quad \exists i : \mathit{LP}(g_i) \mid \mathit{LP}(f)$$

Gröbner Bases

A set of non-zero polynomials $G = \{g_1, \ldots, g_t\}$ contained in an ideal I, is called a **Gröbner basis** for I if and only if for all $f \in I$ such that $f \neq 0$, there exists $i \in \{1, \ldots, t\}$ such that lp(g_i) divides lp(f).

$$G = \text{GröbnerBasis}(I)$$

$$\Longleftrightarrow$$

$$\forall f \in I : f \neq 0, \quad \exists g_i \in G : \text{lp}(g_i) \mid \text{lp}(f)$$

- $f \in I \iff f \xrightarrow{G}_+ 0$
- $f \in I \iff f = \sum_{i=1}^{t} h_i g_i$, with lp($f$) = max (lp($h_i$)lp($f_i$))
- If $G = \{g_1, \ldots, g_t\} \subseteq I$ is a Gröbner Basis for I, then $I = \langle g_1, \ldots, g_t \rangle$

Questions on the Existence of G.B.

- Does a G.B. always exist for any $I \subset k[x_1, \ldots, x_n]$?
- Can it always be computed? Uniquely (up to ordering)?
- What about minimality of the G.B.?
- Study properties of a G.B., and derive algorithms to compute it.

S-Polynomials: Motivation

$I =< f_1, f_2 >,\ f = h_1 f_1 + h_2 f_2.$

- If $LT(f) \in < LT(f_1), LT(f_2) >$, then some $LP(f_i) | LP(f)$
- But what if $LT(f) \notin < LT(f_1), LT(f_2) >$?

$S(f, g) = \frac{L}{lt(f)} \cdot f - \frac{L}{lt(g)} \cdot g$

- $L =$ LCM $(lp(f), lp(g))$

Buchberger's Theorem

Let $G = \{g_1, \ldots, g_t\}$ be a set of non-zero polynomials in $k[x_1, \ldots, x_n]$. Then G is a Grobner basis for the ideal $I =< g_1, \ldots, g_t >$ if and only if for all $i \neq j$

$$S(g_i, g_j) \xrightarrow{G}_+ 0$$

The S-polynomials account for all missing LT(g_i)'s for the Grobner basis computation.

Example:

- $I = \langle yx - y,\ y^2 - x \rangle$
- $GB(I) = \langle yx - y,\ y^2 - x, x^2 - x \rangle$

Relationship to our Problem

We are given $I \subset F_1[x_1, \ldots, x_n]$, $I = \langle f_1, \ldots, f_s \rangle$. We need to find whether or not $V(I) = \emptyset$?

- Consider Ideal generated by constant polynomial $f = 1$, i.e. $I = \langle 1 \rangle$.
- $V(I) = V(f) = V(1)$. Notice $1 = 0$ have no solution, so $V(I) = \emptyset$
- Given polynomial system: $f_1 = f_2 = \cdots = f_s = 0$
- Do they have a common zero?
- No common zero ($V(f_1, \ldots, f_s) = \emptyset$) $\iff$ $1 \in$ G.B.
- But this is true only over **algebraically closed fields**!

What if k isn't closed?

- Let $\overline{k}$ be the algebraic closure of k.
- Every field k is contained in $\overline{k}$, which is algebraically closed. Every element of $\overline{k}$ is the root of a non-zero $f \in k[x]$.
- Let $I = \langle f_1, \ldots, f_s \rangle \in k[x_1, \ldots, x_n]$. Denote variety over extension field $V_{\overline{k}}(I) = \{(a_1, \ldots, a_n) \in \overline{k}^n \mid f_i(a_1, \ldots, a_n) = 0\}$.
- Note: Ideal in k and variety in $\overline{k}$.

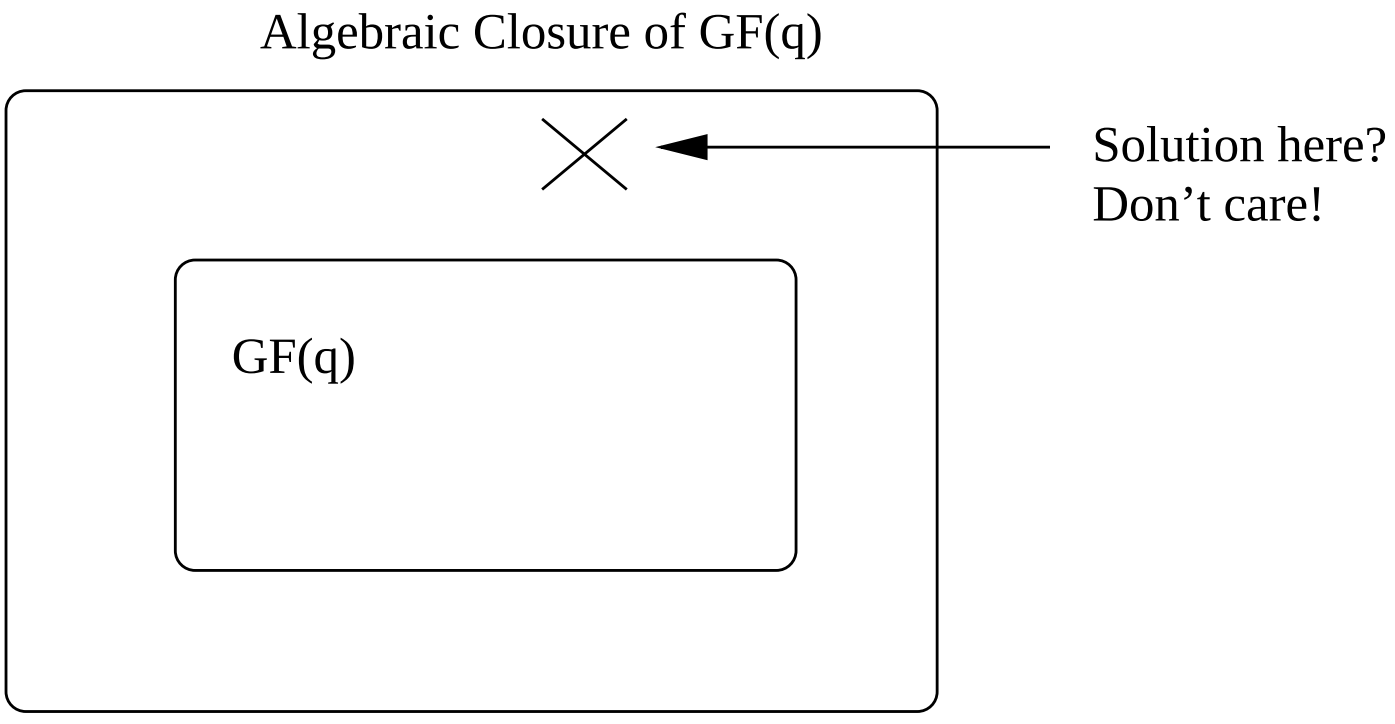
("Weak" Nullstellensatz) Let I be an ideal in $k[x_1, \ldots, x_n]$. Then

$$V_{\overline{k}}(I) = \emptyset \iff I = k[x_1, \ldots, x_n] \iff 1 \in GB(I).$$

Algebraic Closure over GF(2^m)

- Field containment: $GF(2^n) \subset GF(2^m)$ if n divides m.
- $GF(2^2) \subset GF(2^4) \subset GF(2^8) \subset \ldots$
- Algebraic closure of $GF(2^m)$ is an infinite field that is the union of all such fields!
- But, we are looking for solutions in F_q not in $\overline{F_q}$!

Solution in $GF(2^m)$



Vanishing Polynomials in GF(q)

In $F_q[x_1, \ldots, x_n]$:

- $x_i^q = x_i$ in any $GF(q)$.
- $x_i^q - x_i = 0$ in any $GF(q)$.
- $V(x_1^q - x_1, \ldots, x_n^q - x_n) = F_q^n$ the entire affine space!
- $V_{F_q}(I) = V_{\overline{F_q}}(I) \cap F_q$
- $V_{F_q}(I) = V_{\overline{F_q}}(I) \cap V(x_1^q - x_1, \ldots, x_n^q - x_n)$

Given ideal $I = \langle f_1, \ldots, f_s \rangle \subset F_q[x_1, \ldots, x_n]$, and $J = \langle x_i^q - x_i \rangle$, $V_{F_q}(I) = \emptyset \iff 1 \in GB(I, J)$.

Problem: When $q = 2^{256}$, we cannot even declare $x^q - x$ as a polynomial.

Our Experience So Far

- When Spec = Implementation, $V_{\overline{F_q}}(I) = \emptyset$
- When we insert a bug in the design, Gröbner basis computation doesn't terminate
- Behaviour is complementary to SAT:
  - SAT is harder for "No solution" instance; G.B. is easier
  - SAT can find a solution quickly, if one exists, whereas GB computation takes time
- Use a hybrid SAT/GB approach