Notes on Generalized Cofactors

\[ f(x_1, \ldots, x_n) = x_i \cdot f_{x_i} + \overline{x_i} \cdot f_{\overline{x_i}} \]

\( x_i \): literal

Synthesize \( f \) w/ a mux →

\begin{center}
\begin{tikzpicture}
  \node (0) at (0,0) {0};
  \node (1) at (1,0) {1};
  \node (x_i) at (0.5,-1) {x_i};
  \draw (0) -- (x_i) -- (1);
\end{tikzpicture}
\end{center}

Generalize Shannon's Expansion.

Let \( f \) = Boolean function (set of cubes)
\( g \) = Boolean function ("""

\[ f = g \cdot f_g + \overline{g} \cdot f_{\overline{g}} \]

\( f_g, f_{\overline{g}} \): generalized cofactors.

Similar to orthonormal expansion:

Let \( \phi_i \)'s \( i = 1, 2, \ldots, K \) be a set of Boolean functions
such that:
\[
\sum_{i=1}^{K} \phi_i = 1 \quad \text{and} \quad \phi_i \cdot \phi_j = 0 \quad \forall \ i \neq j
\]
Then

\[ f = \phi_1 \cdot f_{\phi_1} + \phi_2 \cdot f_{\phi_2} + \cdots + \phi_k \cdot f_{\phi_k} \]

\[ f_{\phi_i} = \text{generalized cofactors}. \]

**Example:**

\[ f = ab + ac + bc \]

\[ \phi_1 = ab \quad \phi_2 = \overline{a + b} \]

\[ \phi_1 + \phi_2 = 1 \quad \phi_1 \land \phi_2 = 0 \]

So \( \phi_1, \phi_2 \) is a basis.

\[ f = \phi_1 \cdot f_{\phi_1} + \phi_2 \cdot f_{\phi_2} \]

\[ f_{\phi_1} = ab \quad f_{\phi_2} = (ac + bc) \]

or

\[ f_{\bar{\phi}_1} = a \quad f_{\bar{\phi}_2} = (ac + bc) \]

\[ (ab) \cdot (ab) + (\overline{a + b}) \cdot (ac + bc) \]

\[ = ab + \overline{a}bc + \overline{a}c \]

\[ = ab + bc + ac = f \]

\[ a(\overline{ab}) + (\overline{a + b})(ac + bc) \]

\[ = f \]
Generalized cofactors are not unique.
But they satisfy these bounds.

\[
| f \cdot \phi_i | \leq f_{\phi_i} \leq f + \phi_i
\]

Can you prove this?

Demonstrate:
\[
f = ab + ac + bc, \quad \phi_1 = ab, \quad \phi_2 = \overline{a} + b
\]

\[
f \cdot \phi_1 = (ac)(bc) = ab
\]

\[
f + \phi_1 = a b + ac + bc + \overline{a} + b = 1 = \text{everything}
\]

Think of \( f \cdot \phi_1 \) as the \textit{care-set} of the function.

\( f + \phi_1 = \text{case-set} \cup \text{Don't care set} = X's \text{ in } k\text{-map} \)

\( f_{\phi_1} = \text{anything between } f_{\phi_i} \text{ and } f_{\phi_i} \cup X's (D.C.) \)

\( f_{\phi_i} = ab \), or \( f_{\phi_i} = a \) of \( f_{\phi_i} = 1 \) [ they all work ]
Likewise, $\Phi_2 = \overline{a} + b$

\[ f \cdot \Phi_2 = \overline{a} bc + \overline{a} bc \]

\[ f + \Phi_2 = ab + ac + bc = X's \text{ on K-map.} \]

\[ f \Phi_2 = ac + bc. \]

Implications on synthesis:

Given $f$ and corresponding $\Phi$, find $\Phi$ and corresponding $f\Phi$, $\overline{f}\Phi$.

\[ f = \Phi f\Phi + \overline{\Phi} \overline{f}\Phi \]

[Booleen function decomposition]