Combinational Circuit Verification using Strong Nullstellensatz

Overcoming the Complexity of Gröbner Bases for Efficient Verification, and Verification of Integer Multipliers

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What we have learnt so far...

Theorem (Weak NullStellensatz)

Let $\overline{\mathbb{F}}$ be an algebraically closed field. Given ideal $J \subset \overline{\mathbb{F}}[x_1, \ldots, x_n]$, $V_{\overline{\mathbb{F}}}(J) = \emptyset \iff J = \overline{\mathbb{F}}[x_1, \ldots, x_n] \iff 1 \in J \iff reducedGB(J) = \{1\}$.

Theorem (The Strong Nullstellensatz)

Over an algebraically closed field $I(V(J)) = \sqrt{J}$

Verification using Nullstellensatz over $\mathbb{F}_q$

We have two approaches to verify circuits using the Nullstellensatz

- Verify circuits using the miter model
- Construct a miter, and apply the **Weak Nullstellensatz**
- Construct ideal $J_m = \langle f_{spec}, f_1, \ldots, f_s, f_m \rangle$
- Polynomials $f_1, \ldots, f_s$ are the polynomials from the circuit
- J_0 = ideal of all vanishing polynomials
- Circuit $\equiv$ Spec if and only if $GB(J_m + J_0) = \{1\}$.

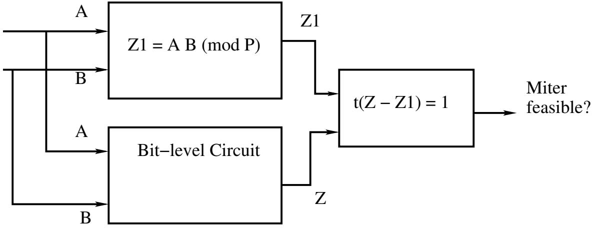


Figure: The equivalence checking setup: miter.

Second Approach to Verification: Strong Nullstellensatz

- The **second** approach is based on Ideal Membership in $I(V(J))$
- Given a spec polynomial f_{spec} and an implementation circuit C
- Derive ideal $J = \langle f_1, \ldots, f_s \rangle$, where $\{f_1, \ldots, f_s\}$ are polynomials from the given circuit C
- It is NOT sufficient to check if $f_{spec} \in J$.
- It is **necessary and sufficient** to check if $f_{spec} \in J + J_0$, where $J_0 =$ ideal of all vanishing polynomials.
- Why? Because f_{spec} may vanish on the variety $V_{\mathbb{F}_q}(J)$
- So, $f \in I(V_{\mathbb{F}_q}(J))$. Remember $I(V(J))$?

$I(V)$

Let $J = \langle f_1, \ldots, f_s \rangle \subset \mathbb{F}[x_1, \ldots, x_n]$. Then:
$I(V(J)) = \{f \in \mathbb{F}[x_1, \ldots, x_n] : f(\mathbf{a}) = 0 \ \forall \mathbf{a} \in V(J)\}$

- $I(V(J))$ is the set of all polynomials that vanish on $V(J)$
- If f vanishes on $V(J)$, then $f \in I(V(J))$

Nullstellensatz over $\mathbb{F}_q$

Theorem (Weak Nullstellensatz over $\mathbb{F}_{2^k}$)

Let ideal $J = \langle f_1, \ldots, f_s \rangle \subset \mathbb{F}_{2^k}[x_1, \ldots, x_n]$ *be an ideal. Let* $J_0 = \langle x_1^{2^k} - x_1, \ldots, x_n^{2^k} - x_n \rangle$ *be the ideal of all vanishing polynomials. Then:*

$$V_{\mathbb{F}_{2^k}}(J) = \emptyset \iff V_{\overline{\mathbb{F}_{2^k}}}(J + J_0) = \emptyset \iff reducedGB(J + J_0) = \{1\}$$

Theorem ($J + J_0$ is radical)

Over Galois fields $\sqrt{J + J_0} = J + J_0$, *i.e.* $J + J_0$ *is a radical ideal.*

Theorem (Strong Nullstellensatz over $\mathbb{F}_q$)

$$I(V_{\mathbb{F}_q}(J)) = I(V_{\overline{\mathbb{F}_q}}(J + J_0)) = \sqrt{J + J_0} = J + J_0$$

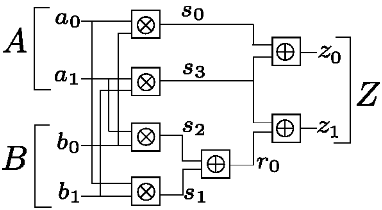
Verification Formulation: The Mathematical Problem

- Given specification polynomial: $f : Z = A \cdot B \pmod{P(x)}$ over $\mathbb{F}_{2^k}$, for given k, and given $P(x)$, s.t. $P(\alpha) = 0$
- Given circuit implementation C
  - Primary inputs: $A = \{a_0, \dots, a_{k-1}\}, B = \{b_0, \dots, b_{k-1}\}$
  - Primary Output $Z = \{z_0, \dots, z_{k-1}\}$
  - $A = a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{k-1}\alpha^{k-1}$
  - $B = b_0 + b_1\alpha + \cdots + b_{k-1}\alpha^{k-1}, \ Z = z_0 + z_1\alpha + \cdots + z_{k-1}\alpha^{k-1}$
- Does the circuit C implement f?

Mathematically:

- Model the circuit (gates) as polynomials: $f_1, \dots, f_s$
  $J = \langle f_1, \dots, f_s \rangle \subset \mathbb{F}_{2^k}[x_1, \dots, x_n]$
- Does f agree with solutions to $f_1 = f_2 = \cdots = f_s = 0$?
- Does f vanish on the Variety $V_{\mathbb{F}_q}(J)$?
- Is $f \in I(V_{\mathbb{F}_q}(J)) = J + J_0$ or is $f \xrightarrow{GB(J+J_0)}_+ 0$?

Example Formulation



Gates as polynomials
$\mathbb{F}_2 \subset \mathbb{F}_{2^k}$:
Ideal J :

$$z_0 = s_0 + s_3; \mapsto f_1 : z_0 + s_0 + s_3$$

$$s_0 = a_0 \cdot b_0; \mapsto f_2 : s_0 + a_0 \cdot b_0$$

$$\vdots$$

$$A + a_0 + a_1\alpha;\ B + b_0 + b_1\alpha;\ Z + z_0 + z_1\alpha$$

Ideal J_0:
$z_0^2 - z_0, s_0^2 - s_0,$

$$\vdots$$

$A^{2^k} - A, B^{2^k} - B,$
$Z^{2^k} - Z$

Complexity of Gröbner Basis

- Complexity of Gröbner basis
  - Degree of polynomials in G is bounded by $2(\frac{1}{2}d^2 + d)^{2^{n-1}}$ [1]
  - Doubly-exponential in n and polynomial in the degree d
- This is the complexity of the GB problem, not of Buchberger's algorithm – that's still a mystery
- For $J \subset \mathbb{F}_q[x_1, \ldots, x_n]$, Complexity $GB(J + J_0) : q^{O(n)}$ (Single exponential)
- Improving Buchberger's algorithm:
  - Improve term ordering (heuristics)
  - Get to all $S(f, g) \xrightarrow{G}_+ 0$ quickly; i.e. arrive at a GB quickly (hard to predict)
  - Improve the implementation of polynomial division; ideas proposed by *Faugére* in the F_4 algorithm

- For $J \subset \mathbb{F}_q[x_1, \ldots, x_n]$, Complexity $GB(J + J_0) : q^{O(n)}$
- GB complexity very sensitive to **term ordering**
- A term order has to be imposed for systematic polynomial computation

Let $f = 2x^2yz + 3xy^3 - 2x^3$

- LEX $x > y > z$: $\quad f = -\mathbf{2x^3} + 2x^2yz + 3xy^3$
- DEGLEX $x > y > z$: $\quad f = \mathbf{2x^2yz} + 3xy^3 - 2x^3$
- DEGREVLEX $x > y > z$: $\quad f = \mathbf{3xy^3} + 2x^2yz - 2x^3$

Recall, S-polynomial depends on term ordering:

$$S(f, g) = \frac{L}{lt(f)} \cdot f - \frac{L}{lt(g)} \cdot g; \qquad L = \text{LCM}(lm(f), lm(g))$$

Effect of Term Orderings on Buchberger's Algorithm

The Product Criteria

If $lm(f) \cdot lm(g) = LCM(lm(f), lm(g))$, then $S(f,g) \xrightarrow{G'}_+ 0$.

LEX: $x_0 > x_1 > x_2 > x_3$

- $f = x_0x_1 + x_2$, $g = x_1x_2 + x_3$
- $lm(f) = x_0x_1$; $\quad lm(g) = x_1x_2$
- $S(f,g) \xrightarrow{G'}_+ x_0x_3 + x_2^2$

LEX: $x_3 > x_2 > x_1 > x_0$

- $f = x_2 + x_0x_1$, $g = x_3 + x_1x_2$
- $lm(f) = x_2$; $\quad lm(g) = x_3$, $S(f,g) \xrightarrow{G'}_+ 0$

"Obviate" Buchberger's algorithm... really?

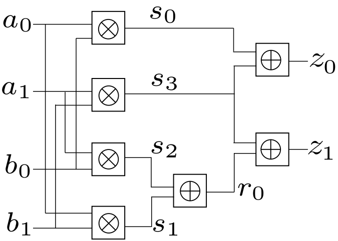
Find a "term order" that makes ALL $\{lm(f),\ lm(g)\}$ relatively prime.

Product Criteria and Gröbner Bases

Recall Buchberger's theorem

The set $G = \{g_1, \ldots, g_t\}$ is a Gröbner basis **iff** for all pairs $(f, g) \in G,\ S(f, g) \xrightarrow{G}_+ 0$

- If we can make leading monomials of all pairs $lm(f), lm(g)$ relatively prime, then all $Spoly(f, g)$ reduce to 0
- This would imply that the polynomials already constitute a Gröbner basis
- No need to compute a GB, may be able to circumvent the GB complexity issues
- Can a term order be derived that makes leading monomials of all polynomials relatively prime?
  - For an "acyclic" circuit, make the gate output variable x_i greater than all variables x_j that are inputs to the gate



$f_1 : s_0 + a_0 \cdot b_0$; $f_2 : s_1 + a_0 \cdot b_1$; $f_3 : s_2 + a_1 \cdot b_0$;

$f_4 : s_3 + a_1 \cdot b_1$; $f_5 : r_0 + s_1 + s_2$; $f_6 : z_0 + s_0 + s_3$

$f_7 : z_1 + r_0 + s_3$; $f_8 : A + a_0 + a_1\alpha$; $f_9 : B + b_0 + b_1\alpha$

$f_{10} : Z + z_0 + z_1\alpha$;

- Perform a Reverse Topological Traversal of the circuit, order the variables according to their reverse topological levels
- LEX with $Z > \{A > B\} > \{z_0 > z_1\} > \{r_0 > s_0 > s_3\} > \{s_1 > s_2\} > \{a_0 > a_1 > b_0 > b_1\}$
- This makes every gate output a leading term, and $\{f_1, \ldots, f_{10}\}$ is a Gröbner basis

This term order also renders a Gröbner Basis of $J + J_0$

Using the Topological Term Order:

- $F = \{f_1, \ldots, f_s\}$ is a Gröbner Basis of $J = \langle f_1, \ldots, f_s \rangle$
- $F_0 = \{x_1^q - x_1, \ldots, x_n^q - x_n\}$ is also a Gröbner basis of J_0 (these polynomials also have relatively prime leading terms)
- But we have to compute a Gröbner Basis of $J + J_0 = \langle f_1, f_2 \ldots, f_s, \;\; x_1^q - x_1, \ldots, x_n^q - x_n \rangle$
- It turns out that $\{f_1, f_2 \ldots, f_s, \;\; x_1^q - x_1, \ldots, x_n^q - x_n\}$ is a Gröbner basis!!
- From our circuit: $f_i = x_i + tail(f_i) = x_i + P$
- Vanishing polynomials $x_i^q - x_i$ with same variable x_i
- Only pairs to consider: $S(f_i, \quad x_i^q - x_i)$ in Buchberger's Algorithm
- All other pairs will have relatively prime leading terms, which will reduce to 0 modulo G

This term order renders a Gröbner basis by construction

So, let us compute $S(f_i = x_i + P,\ \ x_i^q - x_i)$:

$$S(f_i = x_i + P, \quad x_i^q - x_i) = x_i^{q-1}P + x_i$$

$$x_i^{q-1}P + x_i \xrightarrow{x_i+P} x_i^{q-2}P^2 + x_i \xrightarrow{x_i+P} \dots \xrightarrow{x_i+P} P^q - P \xrightarrow{J_0}_+ 0$$

Since $P^q - P$ is a vanishing polynomial, $P^q - P \in J_0$ and $P^q - P \xrightarrow{J_0}_+ 0$

Conclusion: The set of polynomials
$F \cup F_0 = \{f_1, \dots, f_s,\ \ x_i^q - x_i, \dots, x_n^q - x_n\}$ is itself a Gröbner basis due to the reverse topological term order derived from the circuit!

Our Minimal Gröbner Basis

Conclusion:

- Our term order makes $G = \{f_1, \ldots, f_s, x_1^q - x_1, \ldots, x_n^q - x_n\}$ a Gröbner Basis
- This GB$(J + J_0)$ can be further simplified (made minimal)
  - Two types of polynomials: $f_i = x_i + P,\ g_i = x_i^q - x_i$
  - Primary inputs bits are never a leading term of any polynomial
  - Primary inputs are not the output of any gate
- For $x_i \notin$ primary inputs, $f_i = x_i + P$ divides $x_i^q - x_i$; remove $x_i^q - x_i$
- Keep $J_0 = \langle x_i^2 - x_i : x_i \in \text{primary input bits}\rangle$

Our term order makes $G = \{f_1, \ldots, f_s,\ \ x_{PI}^2 - x_{PI}\}$ a minimal Gröbner basis by construction!

Verify the circuit only by a reduction: $f \xrightarrow{G}_+ 0$?

Our Overall Approach

- Given the circuit, perform reverse topological traversal
- Derive the term order to represent the polynomials for every gate, call it the Reverse Topological Term Order (RTTO) $>$
- The set: $\{F, F_0\} = \{f_1, \ldots, f_s, \;\; x_i^2 - x_i : x_i \in X_{PI}\}$ is a minimal Gröbner Basis
- Obtain: $f \xrightarrow{F,F_0}_+ r$
- If $r = 0$, the circuit is verified correct
- If $r \neq 0$, then r contains only the primary input variables
- Any SAT assignment to $r \neq 0$ generates a counter-example
- Counter-example found in no time as r is simplified by Gröbner basis reduction

Move the complexity to that of Polynomial Division

Is this Magic? Or have I told you the full story?

- Reduce x^n modulo $\langle x + P \rangle$, how many cancellations?
  - Requires raising P to the n^{th} power
  - P is the $tail(f_i)$
  - Depending upon n, this can become complicated
- **Reduce** this **minimal** GB $G = \{F, F_0\}$, what does it look like?
  - $f_i = x_i + \text{tail}(f_i)$, where $\text{tail}(f_i) = P(x_j), x_i > x_j$
  - There exists $f_j = x_j + \text{tail}(f_j)$, where $f_j \mid P(x_j)$
  - All non-PI variables x_j can be canceled in this reduction
  - Reduction results in GB G with only primary input variables, potentially explosive

This approach should work for specification polynomials f with low degree terms

Experiments: Correctness Proof, Miter Mastrovito v/s Montgomery Multipliers

Table: Verification Results of SAT, SMT, BDD, ABC.

	Word size of the operands k-bits		
Solver	8	12	16
MiniSAT	22.55	*TO*	*TO*
CryptoMiniSAT	7.17	16082.40	*TO*
PrecoSAT	7.94	*TO*	*TO*
PicoSAT	14.85	*TO*	*TO*
Yices	10.48	*TO*	*TO*
Beaver	6.31	*TO*	*TO*
CVC	*TO*	*TO*	*TO*
Z3	85.46	*TO*	*TO*
Boolector	5.03	*TO*	*TO*
SimplifyingSTP	14.66	*TO*	*TO*
ABC	242.78	*TO*	*TO*
BDD	0.10	14.14	1899.69

Experimental Results: Correctness Proof

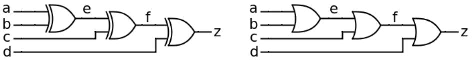
Verify a specification polynomial f against a circuit C by performing the test $f \xrightarrow{J+J_0}_+ 0$?

Table: Verify bug-free and buggy Mastrovito multipliers. SINGULAR computer algebra tool used for division.

Size k-bits	32	64	96	128	160	163
#variables	1155	4355	9603	16899	26243	27224
#polynomials	1091	4227	9411	16643	25923	26989
#terms	7169	28673	64513	114689	179201	185984
Compute-GB:	93.80	*MO*	*MO*	*MO*	*MO*	*MO*
Ours: Bug-free	1.41	112.13	758.82	3054	9361	16170
Ours: Bugs	1.43	114.86	788.65	3061	9384	16368

Why does Compute-GB (SINGULAR) run out of memory?

Limitations of RTTO-based GB-reduction



(a) XOR logic (b) OR logic

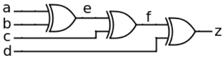
For XOR logic:

$$f_1 : z + f + d \quad f_2 : f + e + c \quad f_3 : e + b + a$$

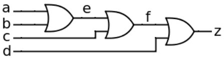
The reduction procedure $z \xrightarrow{f_1, f_2, f_3}_{+} r$ will be computed as follows:

- $z \xrightarrow{z+f+d} f + d$
- $(f + d) \xrightarrow{f+e+c} e + d + c$
- $(e + d + c) \xrightarrow{e+b+a} d + c + b + a$

Limitations of GB-Reduction: OR-gates explode



(c) XOR logic



(d) OR logic

For OR logic:

$$f_1 : z + fd + f + d \quad f_2 : f + ec + e + c \quad f_3 : e + ba + b + a$$

The reduction procedure, $z \xrightarrow{f_1, f_2, f_3}_+ r$ is now computed as:

- $z \xrightarrow{z+fd+f+d} fd + f + d$
- $(fd + f + d) \xrightarrow{f+ec+e+c} f + edc + ed + dc + d;$
  $(f + edc + ed + dc + d) \xrightarrow{f+ec+e+c} edc + ed + ec + e + dc + d + c$
- $(edc + ed + ec + e + dc + d + c) \xrightarrow{e+ba+b+a}_+$
  $dcba + dcb + dca + dba + dc + db + da + d + cba + cb + ca + c + ba + b + a$

Verification of Integer Multipliers

- Use the same ideal membership approach to verify integer multipliers
- Consider a 2-bit (integer multiplier) circuit. Prove that it is an integer multiplier! Or prove that it is buggy.

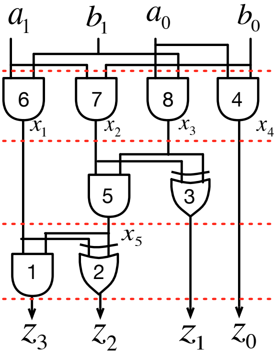


Figure: Integer multiplier circuit

Integer Arithmetic Verification Model

- What is the spec?
- Output word: $z_0 + 2z_1 + 4z_2 + 8z_3$, z_i are bits $\{0, 1\}$
- Input words: $a_0 + 2a_1$, $b_0 + 2b_1$.
- $f_{spec} : z_0 + 2z_1 + 4z_2 + 8z_3 = (a_0 + 2a_1)(b_0 + 2b_1)$
- In polynomial form: $f_{spec} : z_0 + 2z_1 + 4z_2 + 8z_3 - (a_0 + 2a_1)(b_0 + 2b_1)$
- Note f_{spec} has cofficients in $\mathbb{Z}$, but $\mathbb{Z}$ is NOT a field, so we cannot apply Nullstellensatz!
- Trick: Model the problem over $\mathbb{Q}[x_1, \ldots, x_n]$, BUT, use the same RTTO order (important)
- How to model Boolean logic gates over $\mathbb{Q}$?

Model Logic Gates over $\mathbb{Q}$

$$
\begin{aligned}
z = \neg a \mapsto \quad & z = 1 - a \quad \mapsto z - 1 + a \\
z = a \wedge b \mapsto \quad & z = a \cdot b \quad \mapsto z - a \cdot b \\
z = a \vee b \mapsto \quad & z = a + b - a \cdot b \quad \mapsto z - a - b + ab \\
z = a \oplus b \mapsto \quad & z = a + b - 2 \cdot a \cdot b \quad \mapsto z - a - b + 2ab
\end{aligned}
$$

- This requires that every variable take binary values: $a^2 = a$ or $J_0 = \langle a^2 - a, b^2 - b, \ldots, z^2 - z \rangle$
- Construct ideal J from logic gates, add bit-level vanishing polynomials J_0
- What is the leading term of polynomials in J under RTTO?
- Gate output is the leading term, and leading coefficient $= 1$
- Divide by $lc(f) = 1$, division will NEVER produce fractions!

Verification $f_{spec} \pmod{J + J0}$ under RTTO

$$
\begin{aligned}
Z =& 8z_3 + 4z_2 + 2z_1 + z_0 \\
=& \underline{8x_1x_2x_3} + (4x_1 + \underbrace{4x_2x_3} - \underline{8x_1x_2x_3}) \\
& + (2x_2 + 2x_3 - \underbrace{4x_2x_3}) + x_4 \\
=& 4x_1 + 2x_2 + 2x_3 + x_4
\end{aligned}
$$

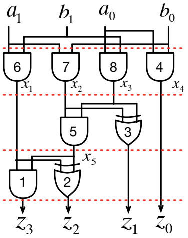


Figure: Integer multiplier circuit

- Ring $R = 0, (z_3, z_2, z_1, z_0, x_5, x_1, x_2, x_3, x_4, a_0, a_1, b_0, b_1), lp;$
- Circuit is an integer multiplier if $f_{spec} \xrightarrow{J+J_0}_{+} 0$.

In Conclusion

The Key to Success in Design Automation

- Build algorithms and techniques on solid theoretical foundations
- Use all of the mathematical tools at your disposal
- Make sure to exploit circuit structure
- Develop domain-specific implementations
- That's what SAT, BDDs, AIGs do too!

T. W. Dube, "The Structure of Polynomial Ideals and Gröbner bases," *SIAM Journal of Computing*, vol. 19, no. 4, pp. 750–773, 1990.