Galois Fields and Hardware Design
Construction of Galois Fields, Basic Properties, Uniqueness, Containment, Closure, Polynomial Functions over Galois Fields

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Lectures conducted Sept 13, 2017 onwards
Agenda

- Introduction to Field Construction
- Constructing $\mathbb{F}_{2^k}$ and its elements
- Addition, multiplication and inverses over GFs
- Conjugates and their minimal polynomials
- GF containment and algebraic closure
- Hardware design over GFs
Integral and Euclidean Domains

Definition

An integral domain $R$ is a set with two operations $(+, \cdot)$ such that:

1. The elements of $R$ form an abelian group under $+$ with additive identity 0.
2. The multiplication is associative and commutative, with multiplicative identity 1.
3. The distributive law holds: $a(b + c) = ab + ac$.
4. The cancellation law holds: if $ab = ac$ and $a \neq 0$, then $b = c$.

Examples: $\mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}_p, \mathbb{F}[x], \mathbb{F}[x, y]$. Finite rings $\mathbb{Z}_n, n \neq p$ are not integral domains.
A Euclidean domain $\mathbb{D}$ is an integral domain where:

1. associated with each non-zero element $a \in \mathbb{D}$ is a non-negative integer $f(a)$ s.t. $f(a) \leq f(ab)$ if $b \neq 0$; and

2. $\forall a, b \ (b \neq 0), \exists (q, r) \text{ s.t. } a = qb + r$, where either $r = 0$ or $f(r) < f(b)$.

- Can apply the Euclid's algorithm to compute $g = \gcd(g_1, \ldots, g_t)$
- $\gcd(a, b, c) = \gcd(\gcd(a, b), c)$
- Then $g = \sum_i u_ig_i$, i.e. $\gcd$ can be represented as a linear combination of the elements
Euclid’s Algorithm

Inputs: Elements \(a, b \in \mathbb{D}\), a Euclidean domain

Outputs: \(g = \gcd(a, b)\)

1: Assume \(a > b\), otherwise swap \(a, b\) \{/* \(\gcd(a, 0) = a\) */\}
2: \textbf{while} \(b \neq 0\) \textbf{do}
3: \hspace{1cm} \(t := b\)
4: \hspace{1cm} \(b := a \mod b\)
5: \hspace{1cm} \(a := t\)
6: \hspace{1cm} \textbf{end while}
7: \textbf{return} \(g := a\)

Algorithm 1: Euclid’s Algorithm
\[ \text{GCD}(84, 54) = 6 \]

\[
\begin{align*}
84 &= 1 \cdot 54 + 30 \\
54 &= 1 \cdot 30 + 24 \\
30 &= 1 \cdot 24 + 6 \\
24 &= 4 \cdot 6 + 0
\end{align*}
\]

**Lemma**

If \( g = \gcd(a, b) \) then \( \exists s, t \) such that \( s \cdot a + t \cdot b = g \).

Unroll Euclid’s algorithm to find \( s, t \). A HW assignment!
Euclidean Domains

- \( \mathbb{D} = \mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}_p \)
- The ring \( \mathbb{F}[x] \) is a Euclidean domain where \( \mathbb{F} \) is any field
- The ring \( R = \mathbb{F}[x, y] \) is NOT a Euclidean domain where \( \mathbb{F} \) is any field
  - For \( x, y \in R \), \( \text{GCD}(x, y) = 1 \), but cannot write
    \[ 1 = f_1(x, y) \cdot x + f_2(x, y)y \]
- \( \mathbb{Z}_{2^k} \) is neither an integral domain nor a Euclidean domain
Let $\mathbb{D}$ be a Euclidean domain, and $p \in \mathbb{D}$ be a prime element. Then $\mathbb{D} \ (\text{mod } p)$ is a field.

- That is why $\mathbb{Z} \ (\text{mod } p)$ is a field
- In $\mathbb{R}[x], x^2 + 1$ is a prime — actually called an irreducible polynomial
- So $\mathbb{R}[x] \ (\text{mod } x^2 + 1)$ is a field and is the field of complex numbers $\mathbb{C}$
- $\mathbb{R}[x] \ (\text{mod } p) = \{ f(x) | \forall g(x) \in \mathbb{R}[x], f(x) = g(x) \ (\text{mod } p) \}$
Let $f, g \in \mathbb{R}[x] \pmod{x^2 + 1}$

- $f$ = remainder of division by $x^2 + 1$, it is linear
- Let $f = ax + b$, $g = cx + d$

\[ f \cdot g = (ax + b)(cx + d) \pmod{x^2 + 1} \]
\[ = acx^2 + (ad + bc)x + bd \pmod{x^2 + 1} \]
\[ = (ad + bc)x + (bd - ac) \text{ after reducing by } x^2 = -1 \]

- Replace $x$ with $i = \sqrt{-1}$, and we get $\mathbb{C}$
- $\mathbb{C}$ is a 2 (=degree($x^2 + 1$)) dimensional extension of $\mathbb{R}$
- Intuitively, that is why $\mathbb{C} \supset \mathbb{R}$ (containment and closure)
Recall from my previous slides:

**From Rings to Fields**

Rings ⊃ Integral Domains ⊃ Unique Factorization Domains ⊃ Euclidean Domains ⊃ Fields

Now you know the reason for this containment
Construct Galois Extension Fields

- $\mathbb{F}_p[x]$ is a Euclidean domain, let $P(x)$ be irreducible over $\mathbb{F}_p$, and let degree of $P(x) = k$
- $\mathbb{F}_p[x] \mod P(x) = \mathbb{F}_{p^k}$, a finite field of $p^k$ elements
- Denote GFs as $\mathbb{F}_q$, $q = p^k$ for prime $p$ and $k \geq 1$
- $\mathbb{F}_{p^k}$ is a $k$-dimensional extension of $\mathbb{F}_p$, so $\mathbb{F}_p \subset \mathbb{F}_{p^k}$
- Our interest $\mathbb{F}_{2^k} = \mathbb{F}_2[x] \mod P(x))$ where $P(x) \in \mathbb{F}_2[x]$ is a degree-$k$ irreducible polynomial
Irreducible polynomials of any degree \( k \) always exist over \( \mathbb{F}_2 \), so \( \mathbb{F}_{2^k} \) can be constructed for arbitrary \( k \geq 1 \)

**Table**: Some irreducible polynomials in \( \mathbb{F}_2[x] \).

<table>
<thead>
<tr>
<th>Degree</th>
<th>Irreducible Polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x; x + 1 )</td>
</tr>
<tr>
<td>2</td>
<td>( x^2 + x + 1 )</td>
</tr>
<tr>
<td>3</td>
<td>( x^3 + x + 1; x^3 + x^2 + 1 )</td>
</tr>
<tr>
<td>4</td>
<td>( x^4 + x + 1; x^4 + x^3 + 1; x^4 + x^3 + x^2 + x + 1 )</td>
</tr>
</tbody>
</table>
\[ \mathbb{F}_{2^k} = \mathbb{F}_2[x] \pmod{P(x)}, \text{ let } \alpha \text{ be a root of } P(x), \text{ i.e. } P(\alpha) = 0 \]

\[ P(x) \text{ has no roots in } \mathbb{F}_2 \text{ (irreducible); root lies in its algebraic extension } \mathbb{F}_{2^k} \]

Any element \( A \in \mathbb{F}_{2^k} \):
\[ A = \sum_{i=0}^{k-1} (a_i \cdot \alpha^i) = a_0 + a_1 \cdot \alpha + \cdots + a_{k-1} \cdot \alpha^{k-1} \text{ where } a_i \in \mathbb{F}_2 \]

The “degree” of \( A < k \)

Think of \( A = \{a_{k-1}, \ldots, a_0\} \) as a bit-vector
Example of $\mathbb{F}_{16}$

- $\mathbb{F}_{2^4}$ as $\mathbb{F}_2[x] \ (\text{mod } P(x))$, where $P(x) = x^4 + x^3 + 1$, $P(\alpha) = 0$
- Any element $A \in \mathbb{F}_{16} = a_3\alpha^3 + a_2\alpha^2 + a_1\alpha + a_0$ (degree $< 4$)

**Table:** Bit-vector, Exponential and Polynomial representation of elements in $\mathbb{F}_{2^4} = \mathbb{F}_2[x] \ (\text{mod } x^4 + x^3 + 1)$

<table>
<thead>
<tr>
<th>$a_3a_2a_1a_0$</th>
<th>Expo</th>
<th>Poly</th>
<th>$a_3a_2a_1a_0$</th>
<th>Expo</th>
<th>Poly</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>0</td>
<td>0</td>
<td>1000</td>
<td>$\alpha^3$</td>
<td>$\alpha^3$</td>
</tr>
<tr>
<td>0001</td>
<td>1</td>
<td>1</td>
<td>1001</td>
<td>$\alpha^4$</td>
<td>$\alpha^3 + 1$</td>
</tr>
<tr>
<td>0010</td>
<td>$\alpha$</td>
<td>$\alpha$</td>
<td>1010</td>
<td>$\alpha^{10}$</td>
<td>$\alpha^3 + \alpha$</td>
</tr>
<tr>
<td>0011</td>
<td>$\alpha^{12}$</td>
<td>$\alpha + 1$</td>
<td>1011</td>
<td>$\alpha^5$</td>
<td>$\alpha^3 + \alpha + 1$</td>
</tr>
<tr>
<td>0100</td>
<td>$\alpha^2$</td>
<td>$\alpha^2$</td>
<td>1100</td>
<td>$\alpha^{14}$</td>
<td>$\alpha^3 + \alpha^2$</td>
</tr>
<tr>
<td>0101</td>
<td>$\alpha^9$</td>
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<td>1101</td>
<td>$\alpha^{11}$</td>
<td>$\alpha^3 + \alpha^2 + 1$</td>
</tr>
<tr>
<td>0110</td>
<td>$\alpha^{13}$</td>
<td>$\alpha^2 + \alpha$</td>
<td>1110</td>
<td>$\alpha^8$</td>
<td>$\alpha^3 + \alpha^2 + \alpha$</td>
</tr>
<tr>
<td>0111</td>
<td>$\alpha^7$</td>
<td>$\alpha^2 + \alpha + 1$</td>
<td>1111</td>
<td>$\alpha^6$</td>
<td>$\alpha^3 + \alpha^2 + \alpha + 1$</td>
</tr>
</tbody>
</table>
Add, Mult in $\mathbb{F}_{2^k}$

**Definition**

The characteristic of a finite field $\mathbb{F}_q$ with unity element 1 is the smallest integer $n$ such that $1 + \cdots + 1$ ($n$ times) $= 0$. 

P. Kalla (Univ. of Utah) $\mathbb{F}_{2^k}$ and Hardware Design / 40
The characteristic of a finite field $\mathbb{F}_q$ with unity element 1 is the smallest integer $n$ such that $1 + \cdots + 1$ ($n$ times) $= 0$.

- What is the characteristic of $\mathbb{F}_{2^k}$? Of $\mathbb{F}_{p^k}$?
Add, Mult in $\mathbb{F}_{2^k}$

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- Characteristic $= 2$ and $p$, respectively, of course!
Add, Mult in $\mathbb{F}_{2^k}$

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\[
\alpha^5 + \alpha^{11} = \alpha^3 + \alpha + 1 + \alpha^3 + \alpha^2 + 1 \\
= 2 \cdot \alpha^3 + \alpha^2 + \alpha + 2 \\
= \alpha^2 + \alpha \quad \text{(as characteristic of $\mathbb{F}_{2^k} = 2$)} \\
= \alpha^{13}
\]
Add, Mult in $\mathbb{F}_{2^k}$

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\]

Addition in $\mathbb{F}_{2^k}$ is Bit-vector XOR operation
Add, Mult in $\mathbb{F}_{2^k}$

\[
\alpha^4 \cdot \alpha^{10} = (\alpha^3 + 1)(\alpha^3 + \alpha) \\
= \alpha^6 + \alpha^4 + \alpha^3 + \alpha \\
= \alpha^4 \cdot \alpha^2 + (\alpha^4 + \alpha^3) + \alpha \\
= (\alpha^3 + 1) \cdot \alpha^2 + (1) + \alpha \quad \text{(as } \alpha^4 = \alpha^3 + 1) \\
= \alpha^5 + \alpha^2 + \alpha + 1 \\
= \alpha^4 \cdot \alpha + \alpha^2 + \alpha + 1 \\
= (\alpha^3 + 1) \cdot \alpha + \alpha^2 + \alpha + 1 \\
= \alpha^4 + \alpha^2 + 1 \\
= \alpha^3 + \alpha^2
\]

Reduce everything \((\text{mod } P(x) = x^4 + x^3 + 1)\), and \(-1 = +1\) in $\mathbb{F}_{2^k}$
Every non-zero element has an inverse

- How to find the inverse of $\alpha$?
- HW for you: think Euclidean algorithm!
- What is the inverse of $\alpha$ in our $\mathbb{F}_{16}$ example?
Vanishing Polynomials of $\mathbb{F}_q$

Lemma

Let $A$ be any non-zero element in $\mathbb{F}_q$, then $A^{q-1} = 1$.

Theorem

[Generalized Fermat's Little Theorem] Given a finite field $\mathbb{F}_q$, each element $A \in \mathbb{F}_q$ satisfies: $A^q \equiv A$ or $A^q - A \equiv 0$

Example

Given $\mathbb{F}_{2^2} = \{0, 1, \alpha, \alpha + 1\}$ with $P(x) = x^2 + x + 1$, where $P(\alpha) = 0$.

$0^{2^2} = 0; \quad 1^{2^2} = 1; \quad \alpha^{2^2} = \alpha \pmod{\alpha^2 + \alpha + 1}$

and

$(\alpha + 1)^{2^2} = \alpha + 1 \pmod{\alpha^2 + \alpha + 1}$
An irreducible poly $P(x)$ is primitive if its root $\alpha$ can generate all non-zero elements of the field.

$\mathbb{F}_q = \{0, 1 = \alpha^{q-1}, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{q-2}\}$

$x^4 + x^3 + 1$ is primitive but $x^4 + x^3 + x^2 + x + 1$ is not

\[
\begin{align*}
\alpha^4 &= \alpha^3 + \alpha^2 + \alpha + 1 \\
\alpha^5 &= \alpha^4 \cdot \alpha \\
&= (\alpha^3 + \alpha^2 + \alpha + 1)(\alpha) \\
&= (\alpha^4) + \alpha^3 + \alpha^2 + \alpha \\
&= (\alpha^3 + \alpha^2 + \alpha + 1) + (\alpha^3 + \alpha^2 + \alpha) \\
&= 1
\end{align*}
\]
Conjugates of $\alpha$

**Theorem**

Let $f(x) \in \mathbb{F}_2[x]$ be an arbitrary polynomial, and let $\beta$ be an element in $\mathbb{F}_{2^k}$ for any $k > 1$. If $\beta$ is a root of $f(x)$, then for any $l \geq 0$, $\beta^{2^l}$ is also a root of $f(x)$. Elements $\beta^{2^l}$ are conjugates of each other.

**Example**

Let $\mathbb{F}_{16} = \mathbb{F}_2[x] \pmod{P(x) = x^4 + x^3 + 1}$. Let $P(\alpha) = 0$. Let us find conjugates of $\alpha$ as $\alpha^{2^l}$.

\[
\begin{align*}
  l = 1: & \quad \alpha^2 \\
  l = 2: & \quad \alpha^4 = \alpha^3 + 1 \\
  l = 3: & \quad \alpha^8 = \alpha^3 + \alpha^2 + \alpha \\
  l = 4: & \quad \alpha^{16} = \alpha \quad \text{(conjugates start to repeat)}
\end{align*}
\]

So $\alpha, \alpha^2, \alpha^3 + 1, \alpha^3 + \alpha^2 + \alpha$ are conjugates of each other.
Get the irreducible polynomial back from conjugates

**Example**

Over $\mathbb{F}_{16} = \mathbb{F}_2[x] \pmod{x^4 + x^3 + 1}$, conjugate elements:

- $\alpha, \alpha^2, \alpha^4, \alpha^8$
- $\alpha^3, \alpha^6, \alpha^{12}, \alpha^{24}$
- $\alpha^7, \alpha^{14}, \alpha^{28}, \alpha^{56}$
- $\alpha^5, \alpha^{10}$

**Minimal Polynomial of an element $\beta$**

Let $e$ be the smallest integer such that $\beta^{2^e} = \beta$. Construct the polynomial $f(x) = \prod_{i=0}^{e-1}(x + \beta^{2^i})$. Then $f(x)$ is an irreducible polynomial, and it is also called the irreducible polynomial of $\beta$. 
Get the irreducible polynomial back from conjugates

Minimal polynomial of any element $\beta$ is: $f(x) = \prod_{i=0}^{e-1}(x + \beta^{2^i})$

Example

Over $\mathbb{F}_{16} = \mathbb{F}_2[x] \pmod{x^4 + x^3 + 1}$, conjugate elements and their minimal polynomials are:

- $\alpha, \alpha^2, \alpha^4, \alpha^8$: $f_1(x) = (x + \alpha)(x + \alpha^2)(x + \alpha^4)(x + \alpha^8) = x^4 + x^3 + 1$
- $\alpha^3, \alpha^6, \alpha^{12}, \alpha^{24}$: $f_2(x) = x^4 + x^3 + x^2 + 1$
- $\alpha^7, \alpha^{14}, \alpha^{28}, \alpha^{56}$: $f_3(x) = x^4 + x + 1$
- $\alpha^5, \alpha^{10}$: $f_4(x) = x^2 + x + 1$

Some observations....

Note that $f_4 = x^2 + x + 1$ is the polynomial used to construct $\mathbb{F}_4$. Also notice that associated with every element in $\mathbb{F}_{2^k}$ is a minimal polynomial and its roots (conjugates), that demonstrate the containment of fields and also the uniqueness of the fields upto the labeling of the elements.
Containment of fields and elements

**Figure:** Containment of fields: $\mathbb{F}_2 \subset \mathbb{F}_4 \subset \mathbb{F}_{16}$

Additive & Multiplicative closure: $\alpha^5 + \alpha^{10} = 1$, $\alpha^5 \cdot \alpha^{10} = 1$. 
Containment and Closure

**Theorem**

\[ F_{2^n} \subset F_{2^m} \text{ if } n \text{ divides } m. \]

**Example:**

\[ F_2 \subset F_{2^2} \subset F_{2^4} \subset F_{2^8} \subset \ldots \]
\[ F_2 \subset F_{2^3} \subset F_{2^6} \subset \ldots \]
\[ F_2 \subset F_{2^5} \subset F_{2^{10}} \subset \ldots \]
\[ F_2 \subset F_{2^7} \subset F_{2^{14}} \subset \ldots \text{ and so on} \]

**Algebraic Closure of \( F_q \)**

The algebraic closure of \( F_{2^k} \) is the union of ALL such fields \( F_{2^n} \) where \( k \mid n \).
Any combinational circuit with $k$-bit inputs and $k$-bit output

- Implements a function $f : \mathbb{B}^k \rightarrow \mathbb{B}^k$
- Can be viewed as a function $f : \mathbb{F}_{2^k} \rightarrow \mathbb{F}_{2^k}$ or $f : \mathbb{Z}_{2^k} \rightarrow \mathbb{Z}_{2^k}$
- Need symbolic representations: view them as polynomial functions

Treat the circuit $f : \mathbb{B}^k \rightarrow \mathbb{B}^k$ as a polynomial function

Please see the last section in my book chapter
Polynomial Functions $f : \mathbb{F}_q \rightarrow \mathbb{F}_q$

- Every function is a polynomial function over $\mathbb{F}_q$

<table>
<thead>
<tr>
<th>${a_2a_1a_0}$</th>
<th>$A$</th>
<th>$\rightarrow$</th>
<th>${z_2z_1z_0}$</th>
<th>$Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>000</td>
<td>0</td>
<td>$\rightarrow$</td>
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<td>011</td>
<td>$\alpha + 1$</td>
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<td>101</td>
<td>$\alpha^2 + 1$</td>
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<td>$\alpha$</td>
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<tr>
<td>110</td>
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<td>$\rightarrow$</td>
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<td>$\alpha + 1$</td>
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<tr>
<td>111</td>
<td>$\alpha^2 + \alpha + 1$</td>
<td>$\rightarrow$</td>
<td>011</td>
<td>$\alpha + 1$</td>
</tr>
</tbody>
</table>
Polynomial Functions \( f : \mathbb{F}_q \rightarrow \mathbb{F}_q \)

- Every function is a polynomial function over \( \mathbb{F}_q \)
- Consider 1-bit right-shift operation \( Z[2:0] = A[2:0] \gg 1 \)

<table>
<thead>
<tr>
<th>( {a_2a_1a_0} )</th>
<th>( A )</th>
<th>( \rightarrow )</th>
<th>( {z_2z_1z_0} )</th>
<th>( Z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>000</td>
<td>0</td>
<td>→</td>
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<td>001</td>
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<td>010</td>
<td>( \alpha )</td>
<td>→</td>
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<td>1</td>
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<tr>
<td>011</td>
<td>( \alpha + 1 )</td>
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<td>100</td>
<td>( \alpha^2 )</td>
<td>→</td>
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<td>→</td>
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<td>( \alpha + 1 )</td>
</tr>
</tbody>
</table>

\[
Z = (\alpha^2 + 1)A^4 + (\alpha^2 + 1)A^2 \text{ over } \mathbb{F}_{2^3} \text{ where } \alpha^3 + \alpha + 1 = 0
\]
Polynomial Functions $f : \mathbb{F}_q \rightarrow \mathbb{F}_q$

Theorem

(From [1]) Any function $f : \mathbb{F}_q \rightarrow \mathbb{F}_q$ is a polynomial function over $\mathbb{F}_q$, that is there exists a polynomial $\mathcal{F} \in \mathbb{F}_q[x]$ such that $f(a) = \mathcal{F}(a)$, for all $a \in \mathbb{F}_q$.

Analyze $f$ over each of the $q$ points, apply Lagrange’s interpolation formula

$$\mathcal{F}(x) = \sum_{n=1}^{q} \frac{\prod_{i \neq n}(x - x_i)}{\prod_{i \neq n}(x_n - x_i)} \cdot f(x_n), \quad (1)$$
Elliptic Curve Cryptography

\[ y^2 + xy = x^3 + ax^2 + b \text{ over } GF(2^k) \]

Compute Slope:

\[ \frac{y_2 - y_1}{x_2 - x_1} \]

Computation of inverses over \( \mathbb{F}_{2^k} \) is expensive
Point addition using Projective Co-ordinates

- Curve: \( Y^2 + XYZ = X^3Z + aX^2Z^2 + bZ^4 \) over \( \mathbb{F}_{2^k} \)
- Let \( (X_3, Y_3, Z_3) = (X_1, Y_1, Z_1) + (X_2, Y_2, 1) \)

\[
\begin{align*}
A &= Y_2 \cdot Z_1^2 + Y_1 \\
B &= X_2 \cdot Z_1 + X_1 \\
C &= Z_1 \cdot B \\
D &= B^2 \cdot (C + aZ_1^2) \\
Z_3 &= C^2 \\
E &= A \cdot C \\
X_3 &= A^2 + D + E \\
F &= X_3 + X_2 \cdot Z_3 \\
G &= X_3 + Y_2 \cdot Z_3 \\
Y_3 &= E \cdot F + Z_3 \cdot G
\end{align*}
\]

No inverses, just addition and multiplication
Multiplication in GF($2^4$)

Input:
\[ A = (a_3 a_2 a_1 a_0) \]
\[ B = (b_3 b_2 b_1 b_0) \]
\[ A = a_0 + a_1 \cdot \alpha + a_2 \cdot \alpha^2 + a_3 \cdot \alpha^3 \]
\[ B = b_0 + b_1 \cdot \alpha + b_2 \cdot \alpha^2 + b_3 \cdot \alpha^3 \]

Irreducible Polynomial:
\[ P = (11001) \]
\[ P(x) = x^4 + x^3 + 1, \quad P(\alpha) = 0 \]

Result:
Output \( G = A \times B \) (mod \( P(x) \))
Multiplication over GF($2^4$)

\[
\begin{array}{cccc}
\times & a_3 & a_2 & a_1 & a_0 \\
& b_3 & b_2 & b_1 & b_0 \\
\hline
a_3 \cdot b_0 & a_2 \cdot b_0 & a_1 \cdot b_0 & a_0 \cdot b_0 \\
a_3 \cdot b_1 & a_2 \cdot b_1 & a_1 \cdot b_1 & a_0 \cdot b_1 \\
a_3 \cdot b_2 & a_2 \cdot b_2 & a_1 \cdot b_2 & a_0 \cdot b_2 \\
a_3 \cdot b_3 & a_2 \cdot b_3 & a_1 \cdot b_3 & a_0 \cdot b_3 \\
s_6 & s_5 & s_4 & s_3 & s_2 & s_1 & s_0 \\
\end{array}
\]

In polynomial expression:
\[
S = s_0 + s_1 \cdot \alpha + s_2 \cdot \alpha^2 + s_3 \cdot \alpha^3 + s_4 \cdot \alpha^4 + s_5 \cdot \alpha^5 + s_6 \cdot \alpha^6
\]

S should be further reduced (mod $P(x)$)
### Multiplication over $\GF(2^4)$

<table>
<thead>
<tr>
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<th>$s_6$</th>
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</table>

$s_4 \cdot \alpha^4 \pmod{\alpha^4 + \alpha^3 + 1} = s_4(\alpha^3 + 1) = s_4 \cdot \alpha^3 + s_4$

$s_5 \cdot \alpha^5 \pmod{\alpha^4 + \alpha^3 + 1} = s_5(\alpha^3 + \alpha + 1) = s_5 \cdot \alpha^3 + s_5 \cdot \alpha + s_5$

$s_6 \cdot \alpha^6 \pmod{\alpha^4 + \alpha^3 + 1} = s_6(\alpha^3 + \alpha^2 + \alpha + 1)$

$= s_6 \cdot \alpha^3 + s_6 \cdot \alpha^2 + s_6 \cdot \alpha + s_6$

$G = g_0 + g_1 \cdot \alpha + g_2 \cdot \alpha^2 + g_3 \cdot \alpha^3$
Montgomery Architecture

Montgomery Multiply: \( F = A \cdot B \cdot R^{-1}, \quad R = \alpha^k \)

- Barrett architectures do not require precomputed \( R^{-1} \)
- We can verify 163-bit circuits, and also catch bugs!
- Conventional techniques fail beyond 16-bit circuits

Figure: Montgomery multiplier over GF(2^k)
Verification: The Mathematical Problem

Let us take verification of GF multipliers as an example:

- **Given specification polynomial:** \( f : Z = A \cdot B \pmod{P(x)} \) over \( \mathbb{F}_{2^k} \), for given \( k \), and given \( P(x) \), s.t. \( P(\alpha) = 0 \)
- **Given circuit implementation** \( C \)
  - Primary inputs: \( A = \{a_0, \ldots, a_{k-1}\} \), \( B = \{b_0, \ldots, b_{k-1}\} \)
  - Primary Output \( Z = \{z_0, \ldots, z_{k-1}\} \)
  - \( A = a_0 + a_1 \alpha + a_2 \alpha^2 + \cdots + a_{k-1} \alpha^{k-1} \)
  - \( B = b_0 + b_1 \alpha + \cdots + b_{k-1} \alpha^{k-1} \), \( Z = z_0 + z_1 \alpha + \cdots + z_{k-1} \alpha^{k-1} \)
- Does the circuit \( C \) correctly compute specification \( f \)?

Mathematically:

- Construct a miter between the spec \( f \) and implementation \( C \)
- Model the circuit (gates) as polynomials \( \{f_1, \ldots, f_s\} \in \mathbb{F}_{2^k}[x_1, \ldots, x_d] \)
- Apply Weak Nullstellensatz
Spec can be a polynomial $f$, or a circuit implementation $C$
Model the miter gate as: $t(X - Y) = 1$, where $t$ is a free variable
Verify a polynomial spec against circuit $C$

\[ Z_1 = A \cdot B \pmod{P} \]

**Figure:** The equivalence checking setup: miter.

- When $Z = Z_1$,  \( t(Z - Z_1) = 1 \) has no solution: infeasible miter
- When $Z \neq Z_1$: let $t^{-1} = (Z - Z_1)$. Then $t \cdot (t^{-1}) = 1$ always has a solution!
- Apply Nullstellensatz over $\mathbb{F}_{2^k}$
Example Implementation Circuit: Mastrovito Multiplier over $\mathbb{F}_4$

Figure: A 2-bit Multiplier

- Write $A = a_0 + a_1\alpha$ as a polynomial $f_A : A + a_0 + a_1\alpha$
- Polynomials modeling the entire circuit: ideal $J = \langle f_1, \ldots, f_{10} \rangle$

- $f_1 : z_0 + z_1\alpha + Z$
- $f_2 : b_0 + b_1\alpha + B$
- $f_3 : a_0 + a_1\alpha + A$
- $f_4 : s_0 + a_0 \cdot b_0$
- $f_5 : s_1 + a_0 \cdot b_1$
- $f_6 : s_2 + a_1 \cdot b_0$
- $f_7 : s_3 + a_1 \cdot b_1$
- $f_8 : r_0 + s_1 + s_2$
- $f_9 : z_0 + s_0 + s_3$
- $f_{10} : z_1 + r_0 + s_3$
Continue with multiplier verification

- So far, ideal $J = \langle f_1, \ldots, f_{10} \rangle$ models the implementation.
- Let polynomial $f : Z - A \cdot B$ denote the spec.
- Miter polynomial $f_m : t \cdot (Z - Z_1) - 1$.
- Update the ideal representation of the miter: $J = J + \langle f, f_m \rangle$.
- Finally: ideal $J = \langle f_1, \ldots, f_{10}, f, f_m \rangle$ represents the miter circuit.
- $J \subseteq \mathbb{F}_{2k}[A, B, Z, Z_1, a_0, a_1, b_0, b_1, r_0, s_0, \ldots, s_3, t]$.
- Verification problem: is the variety $V_{\mathbb{F}_4}(J) = \emptyset$?
- How will we solve this problem?
Theorem (Weak Nullstellensatz over $\mathbb{F}_{2^k}$)

Let ideal $J = \langle f_1, \ldots, f_s \rangle \subset \mathbb{F}_{2^k}[x_1, \ldots, x_n]$ be an ideal. Let $J_0 = \langle x_1^{2^k} - x_1, \ldots, x_n^{2^k} - x_n \rangle$ be the ideal of all vanishing polynomials.

Then:

$$V_{\mathbb{F}_{2^k}}(J) = \emptyset \iff V_{\mathbb{F}_{2^k}}(J + J_0) = \emptyset \iff \text{reducedGB}(J + J_0) = \{1\}$$

Proof:

$$V_{\mathbb{F}_{2^k}}(J) = V_{\mathbb{F}_{2^k}}(J) \cap \mathbb{F}_{2^k}$$

$$= V_{\mathbb{F}_{2^k}}(J) \cap V_{\mathbb{F}_{2^k}}(J_0) = V_{\mathbb{F}_{2^k}}(J) \cap V_{\mathbb{F}_{2^k}}(J_0)$$

$$= V_{\mathbb{F}_{2^k}}(J + J_0)$$

Remember: $V_{\mathbb{F}_q}(J_0) = V_{\mathbb{F}_q}(J_0)$. The variety of $J_0$ does not change over the field or the closure!
Apply Weak Nullstellessatz to the Miter

- Note: Word-level polynomials $f_A : A + a_0 + a_1 \alpha \in \mathbb{F}_{2^k}
- Gate level polynomials $f_4 : s_0 + a_0 \cdot b_0 \in \mathbb{F}_2$
- Since $\mathbb{F}_2 \subset \mathbb{F}_{2^k}$, we can treat ALL polynomials of the miter, collectively, over the larger field $\mathbb{F}_{2^k}$, so $J \subseteq \mathbb{F}_{2^k}[A, B, Z, Z_1, a_0, a_1, \ldots, z_0, z_1]$
- Consider word-level vanishing polynomials: $A^{2^2} - A$
- What about bit-level vanishing polynomials: $a_0^2 - a_0$
- So, $J_0 = \langle W^{2^k} - W, B^2 - B \rangle$, where $W$ are all the word-level variables, and $B$ are all the bit-level variables
- Now compute $G = GB(J + J_0)$. If $G = \{1\}$, the circuit is correct. Otherwise there is definitely a BUG within the field $\mathbb{F}_{2^k}$