Oct 27 Continuing from the previous lecture

We saw: Given ideal $J = \langle f_1, f_2 \rangle$ & polynomial f. Test if $f \in J$?

$$f \xrightarrow[q_1]{f_1} r_1 \xrightarrow[q_2]{f_2} r_2 = 0$$

So $\underbrace{f = q_1 \cdot f_1 + q_2 \cdot f_2}_{f \in \langle f_1, f_2 \rangle \checkmark} + [r_2 = 0]$

But $$f \xrightarrow[q_2]{f_2} r_1 \xrightarrow[q_1]{f_1} r_2 \neq 0$$

$$f = q_1 f_1 + q_2 f_2 + \underline{\underline{r_{2\,\neq 0}}}.$$

$f \notin \langle f_1, f_2 \rangle$? $\underline{\underline{\text{No}}}$

[$f \in \langle f_1, f_2 \rangle$, But division without a Gröbner basis cannot decide ideal membership!]

In our example:

$$f = y^2x - x, \quad f_1 = yx - y, \quad f_2 = y^2 - x$$

$$J = \langle f_1, f_2 \rangle$$

$$f \xrightarrow{f_2} r_1 \xrightarrow{f_1} r_2 = x^2 - x.$$

we know that $f \in J$.

$$f = q_2 \cdot f_2 + q_1 f_1 + x^2 - x$$

$f \in J$, $f_1, f_2 \in J$, so $x^2 - x \in J$

$$x^2 - x = \underbrace{f}_{\in J} - \underbrace{q_2 f_2}_{\in J} - \underbrace{q_1 f_1}_{\in J}$$

$x^2 - x \in J$, but f_1 & f_2 do not divide it.

why?

Term order + division algorithm.

Over $\mathbb{F}[x]$ = Univariate polynomial rings with coefficients from a field

$$J = \langle f_1, \ldots, f_s \rangle = \langle g \rangle$$

Every ideal in $\mathbb{F}[x]$ is generated by only 1 element.

$$\underline{GB(f_1, f_2, \ldots, f_s) = g = GCD(f_1, \ldots f_s)}$$

<u>Proof</u>: Let $f \in J = \langle f_1, \ldots f_s \rangle$

Pick another polynomial $g \in J$ such that $\deg(g) = \text{LEAST}$. i.e. no other polynomial has lower degree in J.

$f(x) \div g(x)$: $\underline{f = p \cdot g + r}$

either $r = 0$ or $\deg(r) < \deg(g)$

$\boxed{\text{not possible}}$

So $\underline{\underline{r = 0}}$

$f = p \cdot g$

$f_1 = p_1 \cdot g$

$f_2 = p_2 \cdot g$

$\Rightarrow$ All poly in J are divisible by g.

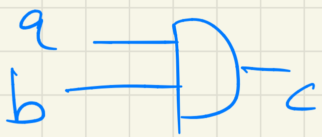
$g = GCD(f_1 \cdots f_s)$

$k\ \mathbb{F}[X]$

$$J = \langle f_1 \cdot f_5 \rangle = \langle g \rangle$$

$$\left.\begin{aligned} f_1(x) &= 0 \\ f_2(x) &= 0 \\ &\ \ \vdots \\ f_5 &= 0 \end{aligned}\right\} \longrightarrow g = 0$$

But Univariate rings are useless for us.



$\mathbb{F}_2[a,b,c] \neq$ PID.

Need G.B.!

GB = generalization of GCD to multivar poly rings!

Study "term orderings" & multivariate division algorithm.

$\Rightarrow$ which will then lead us to GB.

From "An Introduction to Gröbner Bases",
by Adams & Loustaunau, pp. 27, Sec 1.5

Divide $y^2x + 4yx - 3x^2$ by $2y + x + 1$

Term order Deglex $y > x$ $\left[r = f - \frac{Lt(f)}{Lt(g)} \cdot g \right]$

$$\frac{1}{2}yx - \frac{1}{4}x^2$$

$$2y + x + 1 \;\big)\; \cancel{y^2x} + 4yx - 3x^2$$

$$\cancel{y^2x} + \frac{1}{2}yx \qquad + \frac{1}{2}yx^2$$

$$- \qquad - \qquad -$$

Cancel $Lt(f)$ $\longrightarrow$ $0 + \frac{7}{2}yx - 3x^2 - \frac{1}{2}yx^2$

re-order $\Longrightarrow$ $\cancel{-\frac{1}{2}yx^2} + \frac{7}{2}yx - 3x^2$ = 1-step remainder

$$\cancel{-\frac{1}{2}yx^2} \qquad -\frac{1}{4}x^2 \qquad -\frac{1}{4}x^3$$

$$+ \qquad + \qquad +$$

$$\frac{7}{2}yx + \frac{11}{4}x^2 + \frac{1}{4}x^3$$

reorder

$$\frac{1}{4}x^3 + \frac{7}{2}yx - \frac{11}{4}x^2$$

↳ can't be cancelled, move it to the remainder

→ Try to cancel $\frac{7}{2}yx$

continued.....

$$\frac{1}{2}yx - \frac{1}{4}x^2 + \frac{7}{4}x$$

$$2y+x+1 \;\Big|\; y^2x + 4yx - 3x^2$$

$$\frac{1}{4}x^3 + \cancel{\frac{7}{2}yx} - \frac{11}{4}x^2$$

$$\cancel{\overset{+}{-}\frac{7}{2}yx} \quad \overset{+}{-}\frac{7}{4}x^2 \quad \overset{+}{-}\frac{7}{4}x$$

$$\frac{1}{4}x^3 - \frac{9}{2}x^2 - \frac{7}{4}x = r$$

final remainder, as no term in r can be cancelled by $LT(g) = 2y$

Note: $\underset{=3}{\deg(r)} > \underset{=1}{\deg(g)}$

lex $x > y$: $f = -3x^2 + xy^2 + 4xy$

$g = x + 2y + 1$

Now try division of same $f \div g$ but use this term order

$$
\begin{array}{r|l}
 & -3x + y^2 + 10y + 3 \\
\hline
x+2y+1 & \cancel{-3x^2} + xy^2 + 4xy \\
 & \cancel{-3x^2} \qquad -6xy \quad -3x \\
 & + \qquad\qquad +\qquad + \\
\hline
 & \cancel{xy^2} + 10xy + 3x \\
 & \cancel{xy^2} \qquad +2y^3 + y^2 \\
 & - \qquad\qquad - \quad - \\
\hline
 & \cancel{10xy} + 3x - 2y^3 - y^2 \\
 & \cancel{10xy} \qquad +20y^2 + 10y \\
 & - \qquad\qquad - \quad - \\
\hline
 & \cancel{3x} - 2y^3 - 21y^2 - 10y \\
 & \cancel{3x} \qquad +6y + 3 \\
 & - \qquad\qquad - \quad - \\
\hline
 & -2y^3 - 21y^2 - 16y - 3
\end{array}
$$

final remainder.

Change term order $\Rightarrow$ change $Lt(f)$ & $Lt(g)$ $\Rightarrow$ Change quotient and remainders.