Some notes on Boolean Functions and operations:

$$\boxed{f(a,b,c,\ldots) = a\, f_a \underbrace{+}_{\text{Boolean OR}} \bar{a}\, f_{\bar{a}}}$$

This is called the Shannon's Expansion of a Boolean function w.r.t. variable 'a'.

$f_a = f(a=1)$ = positive cofactor of 'f' w.r.t a

$f_{\bar{a}} = f(a=0)$ = negative cofactor.

Significance:- Break down a large 'f' into components (cofactors) and recombine to solve problems.

Example:

$$f = ab + ac + bc$$

$$f_a = f(a=1) = (1)b + (1)c + bc = b + c + bc = b + c.$$

$$f_{\bar{a}} = f(a=0) = 0 + 0 + bc = bc.$$

$$\begin{aligned} f = a f_a + \bar{a} f_{\bar{a}} &= a(b+c) + \bar{a}bc \\ &= ab + ac + \bar{a}bc \\ &= ab + ac + bc \\ &= f \end{aligned}$$

simplify

Operations on co-factors.

Note :- Cofactors f_a, $f_{\bar{a}}$ do not contain the variable 'a'.

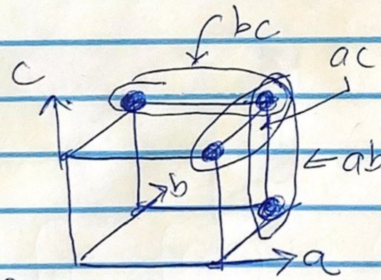
(i) Existential Quantification of f w.r.t 'a'
= smoothing of f w.r.t 'a'

$= \exists_a f = f_a + f_{\bar{a}}$ $+$ = Boolean OR

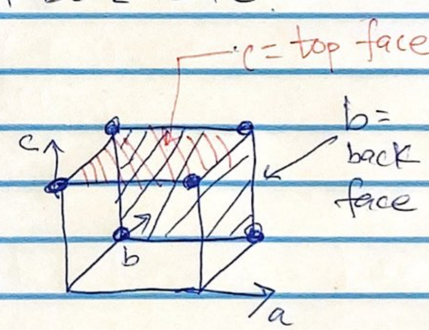
$\exists_a f$ = ~~Largest~~ smallest function, larger than f, contains f, but does not depend on 'a'

Example: $f = ab + ac + bc$. $\left[\begin{array}{l} f_a = b + c \\ f_{\bar{a}} = bc \end{array}\right.$

$\exists_a f = f_a + f_{\bar{a}} = b + c + bc = b + c$



$f = ab + ac + bc$



$\exists_a f = b + c$

Notice = $\boxed{\exists_a f \supseteq f}$ smallest abstraction

Abstraction = over-approximation of 'f'

(ii) Universal Quantification of f w.r.t. 'a'

= Consensus of f w.r.t. 'a'

$$= \forall_a f = f_a \cdot f_{\bar{a}} = f_a \wedge f_{\bar{a}}$$

Boolean AND.

= Largest function, smaller than f, does not contain 'a' in its support.

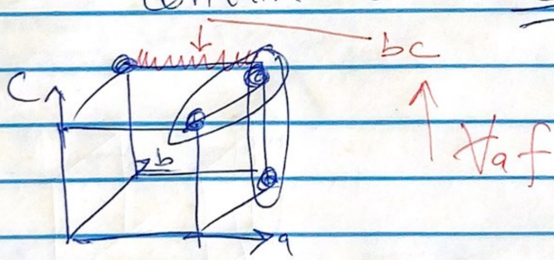
= The component of f that does not contain 'a'.

Example:- $f = ab + ac + bc$

$f_a = b + c$, $f_{\bar{a}} = bc$

$$\forall_a f = f_a \wedge f_{\bar{a}} = (b+c)(bc) = bc + bc = bc$$

The component of f that does not contain 'a' = $\underline{bc}$



f

(iii) Boolean Difference of f wrt 'a'

$$\frac{\partial f}{\partial a} = f_a \oplus f_{\bar a}$$

Example. $f = ab + ac + bc$

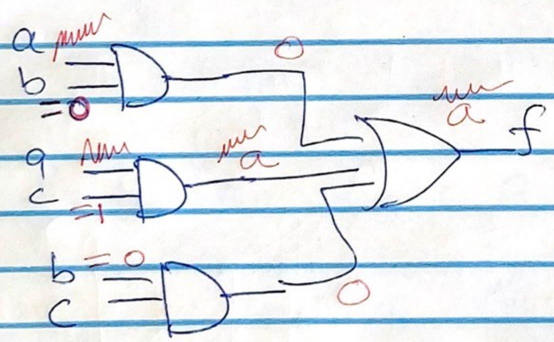
$f_{\bar a} = bc$; $f_a = b + c$

$$\frac{\partial f}{\partial a} = bc \oplus (b+c)$$
$$= (\overline{bc})(b+c) + (bc)(\overline{b+c})$$
$$= (\bar b + \bar c)(b+c) + (bc)(\bar b \cdot \bar c)$$
$$= \bar b c + b \bar c + 0 = \bar b c + b \bar c$$

= Those conditions under which a change in values of 'a' is visible at the output

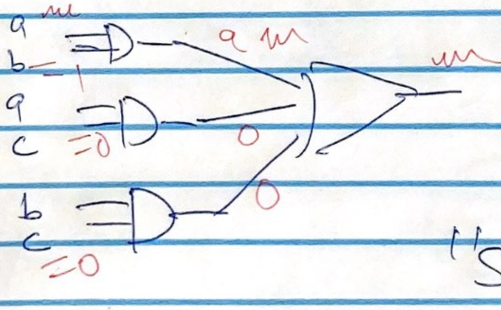
$$\frac{\partial f}{\partial a} = \bar b c + b \bar c$$

take $b=0, c=1$



Those conditions on the "side-inputs" (b, c) where changes in a = visible at output

$b\bar c = b=1, c=0$



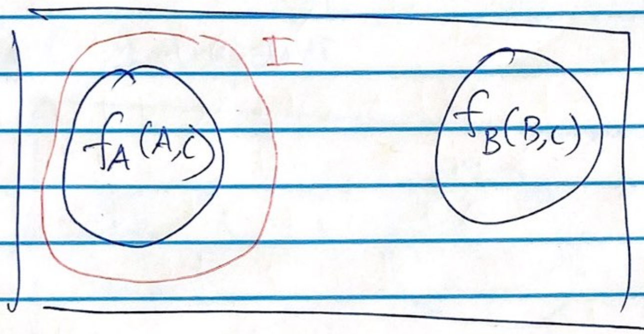
"Use in testing"

"Sensitization"

5

Compute Craig Interpolants using Existential Quantification.

$$f = f_A(A,C) \wedge f_B(B,C) = \phi$$



Interpolant: $I(C)$, $I \supseteq f_A(A,C)$

$$I \wedge f_B(B,C) = \phi$$

$f = f_A \cdot f_B$ $\quad f_A = a_1 a_2 \overline{a_3}$, $f_b = a_2 a_3 \overline{a_4}$

$$f_A \wedge f_B = \phi$$

Smallest Interpolant $\exists_A f_A(A,C)$

Note. $A = a_1$, $C = a_2 a_3$, $B = a_4$.

$$\exists_{a_1} a_1 a_2 \overline{a_3} = (1)\, a_2 \overline{a_3} + (0)\, a_2 \overline{a_3}$$
$$= a_2 \overline{a_3}$$
$$= I_1$$

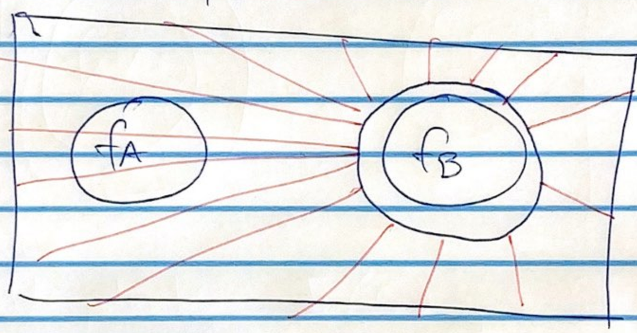
a_1a_2 \ a_3a_4	00	01	11	10
00				
01	X	X		f_B
11	f_A X	f_A X		f_B
10				

$X = I_1 = a_2\overline{a_3}$

= smallest interpolant.

— X — X —

Largest Interpolant



$= \exists_B f_B(B, C) = \underset{= a_4}{\exists_B} (a_2 a_3 \overline{a_4})$

$\exists_{a_4} (a_2 a_3 \overline{a_4}) = a_2 \cdot a_3 = \overline{a_2} + \overline{a_3}$

$= I_2$

a_1a_2 \ a_3a_4	00	01	11	10
00	X	X	X	X
01	X	X		f_B
11	f_A X	f_A X		f_B
10	X	X	X	X

$\overline{a_2}$

$\overline{a_3}$

$I_2(a_2, a_3)$

$I_2 \supset f_A$

$I_2 \wedge f_B = \phi$