Equivalence Checking of Sequential Circuits
Implicit State Enumeration, Image Computations, Traversal of Product Machines

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Nov 29, 2017 - onwards
Define a Mealy Machine as an $n$-tuple:

- $\mathcal{M} = (\sum, O, S, S^0, \Delta, \Lambda)$
- $\sum$: Input label; $O$: output label
- $S$: Set of States; $S^0 \subset S$: set of initial states
- $\Delta: S \times \sum \rightarrow S$: Next State Transition function
- $\Lambda: S \times \sum \rightarrow O$: Output function

![Comb. Logic Diagram]
Two Machines, $M_1$ and $M_2$, are equivalent if

- They are identical; or
- They have identical states but different encoding; or
- $M_1 \subseteq M_2$ or vice-versa; or
- They have different reachable states but same distinguishable states (same condition as above); or
- Different unreachable states, and unreachable states are a don’t care condition

Prove that two machines (sequential circuits) produce the same output response on application of all possible input sequences
Approach

- Build a product machine, and traverse the product machine
- Machine traversal is the core computational engine
- At every step of traversal, see observe of the output responses are the same
- Explicit traversal (DFS) is infeasible
- We use Implicit State Enumeration (BFS traversal) based on Boolean formulas
- Implementation is BDD based (can also use Gröbner)
- First, we will study Implicit State Enumeration, then apply it to product machine
BFS Traversal Algorithm

**Input**: Transition functions $\Delta$, initial state $S^0$

$from^0 = reached = S^0$;

$i = 0$;

repeat

\[
i \leftarrow i + 1;
\]

\[
to^i \leftarrow \text{Img}(\Delta, from^{i-1});
\]

\[
new^i \leftarrow to^i \cap reached;
\]

\[
reached \leftarrow reached \cup new^i;
\]

\[
from^i \leftarrow new^i;
\]

until $new^i == 0$;

return $reached$

- The main computation: $to^i \leftarrow \text{Img}(\Delta, from^{i-1})$
- $\text{Img}(\Delta, from^{i-1})$: The “forward” image of the set $from^{i-1}$ under the transition function $\Delta$
- Let us apply this to a FSM
Algorithm run on the FSM

\( f_{0} = reached = S_{1} \). At iteration \( i = 1 \):

- \( to^{1} \leftarrow Img(\Delta, S_{1}) = \{S_{0}, S_{2}\} \)
- \( new^{1} = \{S_{0}, S_{2}\} \cap \{S_{0}, S_{2}, \ldots S_{5}\} = \{S_{0}, S_{2}\} \)
- \( reached = \{S_{1}, S_{0}, S_{2}\} \), and \( from^{1} = \{S_{0}, S_{2}\} \)

At iteration \( i = 2 \) and \( i = 3 \):

- \( to^{2} = \{S_{0}, S_{2}, S_{3}\}, new^{2} = \{S_{3}\} = from^{2}, reached = \{S_{1}, S_{0}, S_{2}, S_{3}\} \)
- \( to^{3} = \{S_{3}, S_{4}, S_{5}\}, new^{3} = \{S_{4}, S_{5}\} = from^{3}, reached = \{S_{0}, \ldots, S_{5}\} \)
- \( to^{4} = \{S_{2}, S_{5}\}, new^{4} = \emptyset \), algorithm terminates!
BFS traversal without computing new\(^i\) states

**Input:** Transition functions \(\Delta\), initial state \(S^0\)

\[ \text{from}^0 = \text{reached}^0 = S^0; \]
\[ i = 0; \]

**repeat**

\[ i \leftarrow i + 1; \]
\[ \text{to}^i \leftarrow \text{Img}(\Delta, \text{from}^{i-1}); \]
\[ \text{reached}^i \leftarrow \text{reached}^{i-1} \cup \text{to}^i; \]
\[ \text{from}^i \leftarrow \text{reached}^i; \]

**until** \(\text{reached}^i = \text{reached}^{i-1}\)

**return** \(\text{reached}^i\)

This basic approach can be easily implemented using the Gröbner Basis algorithm, details follow in the next few slides.
If a sequential circuit is given:

- The **transition function** \( \Delta \) is given by Boolean equations of the flip-flops of the circuit: \( t_i = \Delta_i(s, x) \)
  - \( t_i \) is the next state variable, \( s \) represents the present state variables and \( x \) represents the input variables
- The **Transition relation of the FSM**:
  \[ T(s, x, t) = \prod_{i=1}^{n} (t_i \oplus \Delta_i) \]
  - \( n \) is the number of flip flops
  - \( \oplus \) is XNOR operation
- Let \( C(s) \) denote the set of initial (or current) states
- Image \( \text{Img}(\Delta, C) = \exists s \exists x [T(s, x, t) \cdot C(s)] \)
  - Remember Smoothing: \( \exists_x f = f_x \lor f_{\overline{x}} \)
  - \( \exists_x f \) is the smallest function that contains \( f \) and makes \( f \) independent of \( x \)

It is not that exotic....
Image of a function

\[ B^n \rightarrow \text{IMAGE}(F, C) \rightarrow B^m \]

\[ N = \text{IMAGE}(F, C) \]
Image Computation on a Circuit

\[ P = x'p'q' + pq \]
\[ Q = p'x + pq' \]
Init State: \( C(s) = p'q' = \{00\} \)

\[ t_1 \oplus \Delta_1 = P \oplus (x'p'q' + pq) \]

\[ t_2 \oplus \Delta_2 = Q \oplus (xp' + pq') \]

\[ T(s, x, t) = [P \oplus (x'p'q' + pq)] \land [Q \oplus (xp' + pq')] \]

Starting from initial state 00, what is the set of next states?

\[
\begin{align*}
to^1 &= \text{Img}(\Delta, \text{from}^0 = C(s)) = \exists p, q, x \ T(p, q, x, P, Q) \land C(s) \\
&= PQ' + P'Q \\
\text{new}^1 &= to^1 \cap \text{reached} \\
&= PQ' + P'Q \\
\text{reached} &= \text{reached} \cup \text{new} = P'Q' + PQ' + P'Q = P' + Q'
\end{align*}
\]

In the next iteration: \( C(s) = \text{from}^1 = \text{new}^1 = PQ' + P'Q \), continue...
BFS Traversal is the core of Sequential Verification

Given $\mathcal{M}_1 = (\sum, O, S^1, S^1_0, \Delta_1, \Lambda_1)$; $\mathcal{M}_2 = (\sum, O, S^2, S^2_0, \Delta_2, \Lambda_2)$

- Build a product machine $\mathcal{M}_1 \times \mathcal{M}_2 = (\sum, O^{12}, S^{12}, S^{12}_0, \Delta_{12}, \Lambda_{12})$
Product Machine

Given $\mathcal{M}_1 = (\sum, O, S^1, S^1_0, \Delta_1, \Lambda_1)$; $\mathcal{M}_2 = (\sum, O, S^2, S^2_0, \Delta_2, \Lambda_2)$

- Build a product machine $\mathcal{M}_1 \times \mathcal{M}_2 = (\sum, O^{12}, S^{12}, S^{12}_0, \Delta_{12}, \Lambda_{12})$, where:
  - $s^{12} = (s^1, s^2) \in S^{12}$: Concatenation of states

\[
\Delta_{12} \equiv \Delta_{12}(s^{12}, x) : (S^1 \times S^2) \times \sum \rightarrow (S^1 \times S^2) \\
\implies ((S^1 \times \sum) \rightarrow S^1) \land ((S^2 \times \sum) \rightarrow S^2) \\
\implies (\Delta_1(s^1, x), \Delta_2(s^2, x))
\]

- $z^{12} = \Lambda_{12}(s^{12}, x) : (S^1 \times S^2) \times \sum \rightarrow \{0, 1\}$, where:

\[
\Lambda_{12}(s^{12}, x) = \begin{cases} 
1 & \text{if } \Lambda_1 = \Lambda_2 \\
0 & \text{otherwise}
\end{cases}
\]
In other words...

The product machine has:
- States that are concatenation of the states of the original machine
- Transition relation is also a concatenation (or conjunction) of the transition relations of the original machines

Verification Problem
Find a sequence of inputs that distinguishes the initial states of the individual machines.

Verification Approach
Compute Next States independently and concatenate the results to form the next state of the product. The outputs specified for the two corresponding transitions are then compared. If equal, then the product machine outputs a 1, otherwise 0 (BUG!)
FSM Verify Example

Build the Product Machine.....
Input: $\sum, \Delta_1, \Delta_2, \Lambda_1, \Lambda_2$, initial states $S_0^1, S_0^2$
from$^0 = \text{reached} = \text{new}^0 = (S_0^1, S_0^2)$; $i = 0$

repeat

\[ i \leftarrow i + 1; \]
\[ \text{to}^i \leftarrow \text{Img}(\Delta_{12}, \text{from}^{i-1}); \]
\[ \text{new}^i \leftarrow \text{to}^i \cap \text{reached}; \]
for each $S^{12} \in \text{new}^i$ do

\[ \text{for each } x \in \sum \text{ do} \]
\[ \text{if } \Lambda_{12}(s, x) = 0 \text{ then} \]
\[ \text{return } \text{FALSE}; \]
\[ \text{end} \]
\[ \text{end} \]
end

\[ \text{from}^i \leftarrow \text{new}^i; \]
\[ \text{reached} \leftarrow \text{reached} \cup \text{new}^i; \]
until $\text{new}^i \neq 0$;