Constraint optimization

A general formula for these problems is

$$\min_{x \in \mathbb{R}^n} f(x)$$

s.t. $c_i(x) = 0, \ i \in E$

$c_i(x) \geq 0, \ i \in I$

where $f$ and the functions $c_i$ are all smooth, real valued functions on a subset of $\mathbb{R}^n$, and $E$ and $I$ are two finite sets of indices.

- $f$ - objective function
- $c_i (i \in E)$ - equality constraints
- $c_i (i \in I)$ - inequality constraints.

We define the feasible set of points $x$ that satisfy the constraints

$$S_L = \{ x \mid c_i(x) = 0, \ i \in E \} . c_i(x) \geq 0, \ i \in I \}$$

Simplified as

$$\min_{x \in S_L} f(x)$$
• A vector \( x^* \) is a **local solution** if \( x^* \in \Omega \) and there is a neighborhood \( N \) of \( x^* \) such that \( f(x) \geq f(x^*) \) for all \( x \in N \cap \Omega \)

\[ N(x^*) \text{ s.t. } f(x) \geq f(x^*) \quad \forall x \in N \cap \Omega \]

• A vector \( x^* \) is a **strict local solution** (aka **strong local solution**) if \( x^* \in \Omega \) and there is a neighborhood \( N \) of \( x^* \) such that \( f(x) > f(x^*) \) for all \( x \in N \cap \Omega \) with \( x \neq x^* \)

\[ N(x^*) \mid f(x) > f(x^*) \quad \forall x \in N \cap \Omega \mid x \neq x^* \]

• A point \( x^* \) is an **isolated local solution** if \( x^* \in \Omega \) and there is a \( N \) of \( x^* \) such that \( x^* \) is the only local solution in \( N \cap \Omega \).

\[ N(x^*) \mid x^* \in N \cap \Omega \]
Linear Programming

- Linear objective function and linear constraints, which may include both equalities and inequalities.

- The feasible set is a polytope (a convex, connected set with flat polygonal faces).

- Contours of objective function are planar.

- Depicts a linear program in 2-D space, in which the contours of the objective function are indicated by dotted lines.

- The problem has no solution, if the feasible set is empty (infeasible case).

- The problem has "", if the function is unbounded below on the feasible region (unbounded case).
Linear programs are usually stated and analyzed in the following standard form:

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

where \( c \) and \( x \) are vectors in \( \mathbb{R}^n \), \( b \) is a vector in \( \mathbb{R}^m \), and \( A \) is an \( m \times n \) matrix with rank \( m \) (i.e. \( m \leq n \) and linear independent).

A simple device can be used to transform any linear program to standard form:

Ex.

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax \leq b \\
& \quad (\text{i.e. no bounds on } x)
\end{align*}
\]

Now we can convert the inequality constraint to equalities by introducing a vector of slack variables \( z \) to make

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax + z = b, \quad z \geq 0
\end{align*}
\]
What's wrong with this form?

Not all variables are constrained to be nonnegative.

We can solve this by splitting \( x \) into its nonnegative and nonpositive parts,

\[
x = x^+ - x^-
\]

where \( x^+ = \max(x, 0) \geq 0 \)

\( x^- = \max(-x, 0) \geq 0 \)

Now the problem becomes:

\[
\begin{bmatrix}
\min \\
[x^+, x^-, z]^T
\end{bmatrix}
\begin{bmatrix}
C \\
-C \\
0
\end{bmatrix}
\begin{bmatrix}
x^+ \\
x^- \\
z
\end{bmatrix}
\]

s.t. \( [A -A I_z] \begin{bmatrix} x^+ \\ x^- \\ z \end{bmatrix} = b \)

\[
\begin{bmatrix}
x^+ \\
x^- \\
z
\end{bmatrix} \geq 0
\]

\hspace{2cm} \text{Standard Form}
Inequality constraints of the form $x \leq u$ or $Ax \geq b$ always can be converted to equality constraints by adding or subtracting slack variables.

\[ x \leq u \iff x + w = u, \quad w \geq 0 \]
\[ Ax \geq b \iff Ax - y = b, \quad y \geq 0 \]

Add (Slack), Subtract (Surplus)

\[ (x, z) \text{ primal variables} \]

We can also convert a $\max c^T x$ to a $\min (-c)^T x$.

Standard Form (Primal),

\[ \max b^T \lambda \\
\text{s.t. } A^T \lambda \leq C \quad (\text{dual}) \]

By

\[ \max b^T \lambda \]
\[ \text{s.t. } A^T \lambda + s = c \]
\[ s \geq 0 \]

\((x, s)\) dual variables.
Strong Duality Theorem

1) If either problem has a (finite) solution, then so does the other, and the objective values are equal.

2) If either problem is unbounded, then the other problem is infeasible.

(Fundamental Theorem of Linear Programming)
- If primal has a nonempty feasible region, then there is at least one basic feasible point.
- If primal has solution, then at least one such solution is a basic optimal point.
- If primal is feasible and bounded, then it has an optimal solution.

5

All basic feasible points for primal are vertices of the feasible polytope $\mathbb{R}^n \setminus \{ A x = b, x \geq 0 \}$ and vice versa.
The simplex method's strategy of examining only basic feasible points will converge to a solution of the primal only if:

a) the problem has basic feasible points

b) at least one such point is a basic optimal point, that is, a solution of the primal that is also a basic feasible point.

There are a number of variants of the simplex method.

**Generalization:**

Proceed from one basic feasible solution to another, in such a way as to continually decrease the value of the objective function until a minimum is reached.
Simplex Logic

Compute initial bfs

Current bfs optimal

Yes → Done

No → Compute better bfs
Example:

\[
\begin{align*}
\text{max} \quad & Z = X_1 + 3X_2 \\
\text{s.t.} \quad & X_1 + X_2 \leq 10 \\
& 5X_1 + 2X_2 \leq 20 \\
& X_1 + 2X_2 \leq 36 \\
& X_1, X_2 \geq 0
\end{align*}
\]

Slack Variables: \((X_3, X_4, X_5)\)

\[
\begin{align*}
X_1 + X_2 + X_3 &= 10 \\
5X_1 + 2X_2 + X_4 &= 20 \\
X_1 + 2X_2 + X_5 &= 36
\end{align*}
\]

\[
Z = X_1 + 3X_2 \rightarrow -X_1 - 3X_2 + Z = 0
\]

Initial Simplex tableau

\[
\begin{array}{cccccc|c}
1 & 1 & 1 & 0 & 0 & 0 & 10 \\
0 & 2 & 0 & 1 & 0 & 0 & 20 \\
0 & 2 & 0 & 0 & 1 & 0 & 36 \\
-1 & -3 & 0 & 0 & 0 & 1 & 0
\end{array}
\]

Pivot column since \(-3\) is most negative indicator
Quotients: \( \frac{10}{1} = 10 \) \( \frac{20}{2} = 10 \) \( \frac{36}{2} = 18 \)

Since we have two equally small quotients, we can choose either entry in column 2 to be the pivot.

Let's choose the number 1 in column 2 to be the pivot.

\[
\begin{align*}
\text{r}_2 & \rightarrow -2\text{r}_1 + \text{r}_2 \\
1 & | \ 1 & 1 & 0 & 0 & 0 & 10 \\
3 & | \ 0 & -2 & 0 & 0 & 0 \\
1 & | \ 2 & 0 & 0 & 1 & 0 & 36 \\
\hline
-1 & | -3 & 0 & 0 & 0 & 0
\end{align*}
\]

\[
\begin{align*}
\text{r}_3 & \rightarrow -2\text{r}_1 + \text{r}_3 \\
1 & | \ 1 & 1 & 0 & 0 & 0 & 10 \\
3 & | \ 0 & -2 & 1 & 0 & 0 \\
\hline
-1 & | -10 & 0 & 10 & 16 \\
-1 & | -3 & 0 & 0 & 0 & 0 \\
\hline
-1 & | -3 & 0 & 0 & 0 & 0
\end{align*}
\]

\[
\begin{align*}
\text{r}_4 & \rightarrow 3\text{r}_1 + \text{r}_4 \\
1 & | \ 1 & 1 & 0 & 0 & 0 & 10 \\
3 & | \ 0 & -2 & 1 & 0 & 0 \\
\hline
-1 & | -10 & 0 & 10 & 16 \\
-1 & | -3 & 0 & 0 & 0 & 0 \\
\hline
0 & | 3 & 0 & 0 & 1 & 30
\end{align*}
\]
Since there are no negatives in the last row, we are done pivoting.

\[ 2x_1 + 3x_3 + z = 30 \]

\[ \Rightarrow z = 30 - 2x_1 - 3x_3 \]

\underline{Basic variables are} \[ x_2, x_4, x_5, \text{ and } z \]

\underline{Non basic variables are} \[ x_1, x_3 \]

Solution is found by setting the \underline{nonbasic variables equal to zero,

\[ x_1 = 0, x_3 = 0 \]

Then from final tableau we have

\[ x_1 + x_2 + x_3 = 10 \]
\[ 3x_1 - 2x_3 + x_4 = 0 \]
\[ -x_1 - 2x_3 + x_5 = 16 \]

\[ \Rightarrow x_2 = 10, x_3 = 0, x_5 = 16 \]

and max is \( z = 30 \) (right lower corner of tableau)
Simplex Method

1. Convert each constraint to an equation by adding slack variables.

2. Set up the initial tableau.

3. Locate the most negative indicator. This indicator determines the pivot column.

4. Use the positive entries in the pivot column to form the necessary quotients for determining the pivot. If there are no positive entries, no maximum solution exists. If two quotients are equally the smallest, let either determine the pivot.

5. Divide the pivot row by the pivot to change the pivot to 1. Then use row operations to change all other entries in the pivot column to 0.

6. If the indicators are all positive or zero, you have found the final tableau.

7. Set each nonbasic variable to 0 and solve the system for the basic variables. The max is in lower right.
Quadratic Programming

The general QP can be stated as

$$\min_{x} \; q(x) = \frac{1}{2} x^T G x + x^T c$$

s.t.  \( A x = b \) \( i \in \mathcal{E} \) (Equality constrained)
\( A x \geq b \) \( i \in \mathcal{I} \) (Inequality constrained)

where \( G \) is symmetric \( n \times n \)
\( \mathcal{E}, \mathcal{I} \) finite sets
\( (c, x, A), i \in \mathcal{E} \cup \mathcal{I} \)

Quadratic programs can always be solved (or shown infeasible) in a finite amount of computation. Depending on problem, strongly.

Convex QP \( \Rightarrow G \) is pos. semi-def. \( x^T M x \geq 0 \)

This case is often similar in difficulty to LP.

Strictly convex QP \( \Rightarrow G \) is pos. def. \( x^T M x > 0 \)

Non convex QP \( \Rightarrow G \) is indefinite
Solution Methods Include

- Interior point
- Active set
- Augmented Lagrangian
- Conjugate gradient
- Extensions of the Simplex Method