6-2 ONE FUNCTION OF TWO RANDOM VARIABLES

Given two random variables **x** and **y** and a function $g(x, y)$, we form a new random variable **z** as

$$z = g(\mathbf{x}, \mathbf{y}) \tag{6-36}$$

Given the joint p.d.f. $f_{xy}(x, y)$, how does one obtain $f_z(z)$, the p.d.f. of **z**? Problems of this type are of interest from a practical standpoint. For example, a received signal in a communication scene usually consists of the desired signal buried in noise, and this formulation in that case reduces to $\mathbf{z} = \mathbf{x} + \mathbf{y}$. It is important to know the statistics of the incoming signal for proper receiver design. In this context, we shall analyze problems of the type shown in Fig. 6-6. Referring to (6-36), to start with,

$$F_z(z) = P\{\mathbf{z}(\xi) \le z\} = P\{g(\mathbf{x}, \mathbf{y}) \le z\} = P\{(\mathbf{x}, \mathbf{y}) \in D_z\}$$

$$= \iint_{x, y \in D_z} f_{xy}(x, y)\, dx\, dy \tag{6-37}$$

where D_z in the xy plane represents the region where the inequality $g(x, y) \le z$ is satisfied (Fig. 6-7).
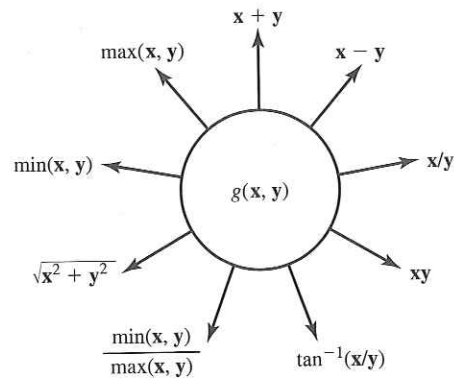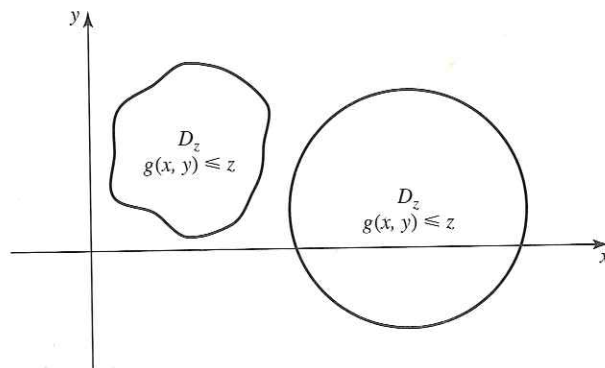

FIGURE 6-6

FIGURE 6-7

Note that D_z need not be simply connected. From (6-37), to determine $F_z(z)$ it is enough to find the region D_z for every z, and then evaluate the integral there.

We shall illustrate this method to determine the statistics of various functions of **x** and **y**.

$z = x + y$

▶ Let $\mathbf{z} = \mathbf{x} + \mathbf{y}$. Determine the p.d.f. $f_z(z)$.

From (6-37),

$$F_z(z) = P\{\mathbf{x} + \mathbf{y} \le z\} = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{z-y} f_{xy}(x, y)\, dx\, dy \tag{6-38}$$

since the region D_z of the xy plane where $x + y \le z$ is the shaded area in Fig. 6-8 to the left of the line $x + y \le z$. Integrating over the horizontal strip along the x axis first (inner integral) followed by sliding that strip along the y axis from $-\infty$ to $+\infty$ (outer integral) we cover the entire shaded area.

We can find $f_z(z)$ by differentiating $F_z(z)$ directly. In this context it is useful to recall the differentiation rule due to Leibnitz. Suppose

$$F_z(z) = \int_{a(z)}^{b(z)} f(x, z)\, dx \tag{6-39}$$

Then

$$f_z(z) = \frac{dF_z(z)}{dz} = \frac{db(z)}{dz} f(b(z), z) - \frac{da(z)}{dz} f(a(z), z) + \int_{a(z)}^{b(z)} \frac{\partial f(x, z)}{\partial z}\, dx \tag{6-40}$$

Using (6-40) in (6-38) we get

$$f_z(z) = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial z} \int_{-\infty}^{z-y} f_{xy}(x, y)\, dx \right) dy$$

$$= \int_{-\infty}^{\infty} \left(1 \cdot f_{xy}(z - y, y) - 0 + \int_{-\infty}^{z-y} \frac{\partial f_{xy}(x, y)}{\partial z} \right) dy$$

$$= \int_{-\infty}^{\infty} f_{xy}(z - y, y)\, dy \tag{6-41}$$

Alternatively, the integration in (6-38) can be carried out first along the y axis followed by the x axis as in Fig. 6-9 as well (see problem set).
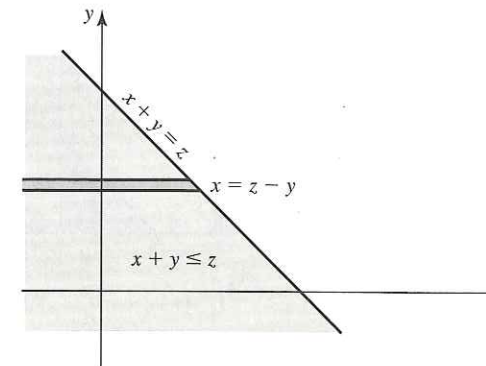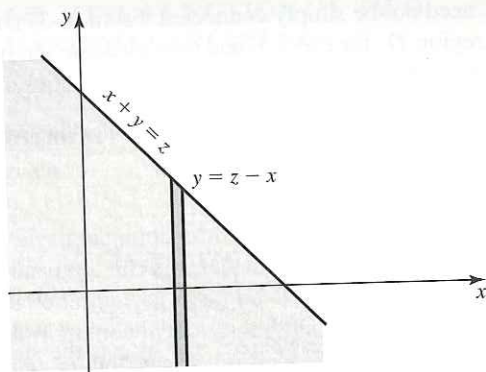

FIGURE 6-8

FIGURE 6-9

If **x** and **y** are independent, then

$$f_{xy}(x, y) = f_x(x) f_y(y) \tag{6-42}$$

and inserting (6-42) into (6-41) we get

$$f_z(z) = \int_{x=-\infty}^{\infty} f_x(z - y) f_y(y)\, dy = \int_{x=-\infty}^{\infty} f_x(x) f_y(z - x)\, dx \tag{6-43}$$

This integral is the convolution of the functions $f_x(z)$ and $f_y(z)$ expressed two different ways. We thus reach the following conclusion: If two random variables are *independent*, then the density of their sum equals the convolution of their densities.

As a special case, suppose that $f_x(x) = 0$ for $x < 0$ and $f_y(y) = 0$ for $y < 0$, then we can make use of Fig. 6-10 to determine the new limits for D_z.

In that case

$$F_z(z) = \int_{y=0}^{z} \int_{x=0}^{z-y} f_{xy}(x, y)\, dx\, dy$$

or

$$f_z(z) = \int_{y=0}^{z} \left(\frac{\partial}{\partial z} \int_{x=0}^{z-y} f_{xy}(x, y)\, dx \right) dy$$

$$= \begin{cases} \int_0^z f_{xy}(z - y, y)\, dy & z > 0 \\ 0 & z \le 0 \end{cases} \tag{6-44}$$
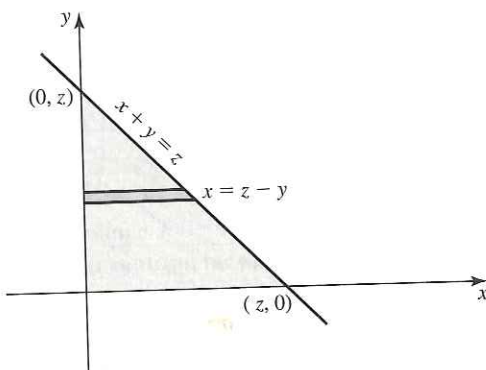

FIGURE 6-10

On the other hand, by considering vertical strips first in Fig. 6-10, we get

$$F_z(z) = \int_{x=0}^{z} \int_{y=0}^{z-x} f_{xy}(x, y)\, dy\, dx$$

or

$$f_z(z) = \int_{x=0}^{z} f_{xy}(x, z - x)\, dx$$

$$= \begin{cases} \int_0^z f_x(x) f_y(z - x)\, dx & z > 0 \\ 0 & z \le 0 \end{cases} \tag{6-45}$$

if **x** and **y** are independent random variables. ◀

EXAMPLE 6-7 ▶ Suppose **x** and **y** are independent exponential random variables with common parameter λ. Then

$$f_x(x) = \lambda e^{-\lambda x} U(x) \qquad f_y(y) = \lambda e^{-\lambda y} U(y) \tag{6-46}$$

and we can make use of (6-45) to obtain the p.d.f. of $z = x + y$.

$$f_z(z) = \int_0^z \lambda^2 e^{-\lambda x} e^{-\lambda(z-x)}\, dx = \lambda^2 e^{-\lambda z} \int_0^z dx$$

$$= z\lambda^2 e^{-\lambda z} U(z) \tag{6-47}$$

As Example 6-8 shows, care should be taken while using the convolution formula for random variables with finite range. ◀

EXAMPLE 6-8 ▶ **x** and **y** are independent uniform random variables in the common interval (0, 1). Determine $f_z(z)$, where $z = x + y$. Clearly,

$$z = x + y \Rightarrow 0 < z < 2$$

and as Fig. 6-11 shows there are two cases for which the shaded areas are quite different in shape, and they should be considered separately.
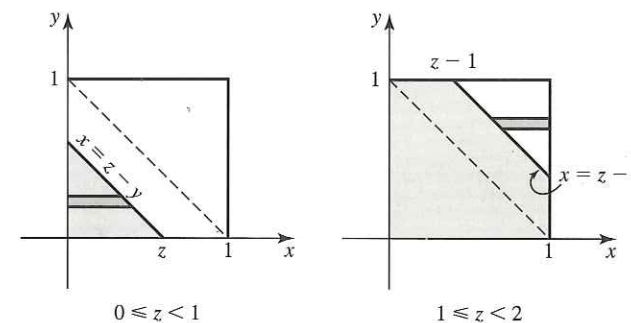

$$0 \le z < 1 \qquad\qquad 1 \le z < 2$$

FIGURE 6-11

For $0 \le z < 1$,

$$F_z(z) = \int_{y=0}^{z} \int_{x=0}^{z-y} 1 \, dx \, dy = \int_{y=0}^{z} (z-y) \, dy = \frac{z^2}{2} \qquad 0 < z < 1 \qquad (6\text{-}48)$$

For $1 \le z < 2$, notice that it is easy to deal with the unshaded region. In that case,

$$F_z(z) = 1 - P\{\mathbf{z} > z\} = 1 - \int_{y=z-1}^{1} \int_{x=z-y}^{1} 1 \, dx \, dy$$

$$= 1 - \int_{y=z-1}^{1} (1 - z + y) \, dy = 1 - \frac{(2-z)^2}{2} \qquad 1 \le z < 2 \qquad (6\text{-}49)$$

Thus

$$f_z(z) = \frac{dF_z(z)}{dz} = \begin{cases} z & 0 \le z < 1 \\ 2 - z & 1 \le z < 2 \end{cases} \qquad (6\text{-}50)$$

By direct convolution of $f_x(x)$ and $f_y(y)$, we obtain the same result as above. In fact, for $0 \le z < 1$ (Fig. 6-12a)

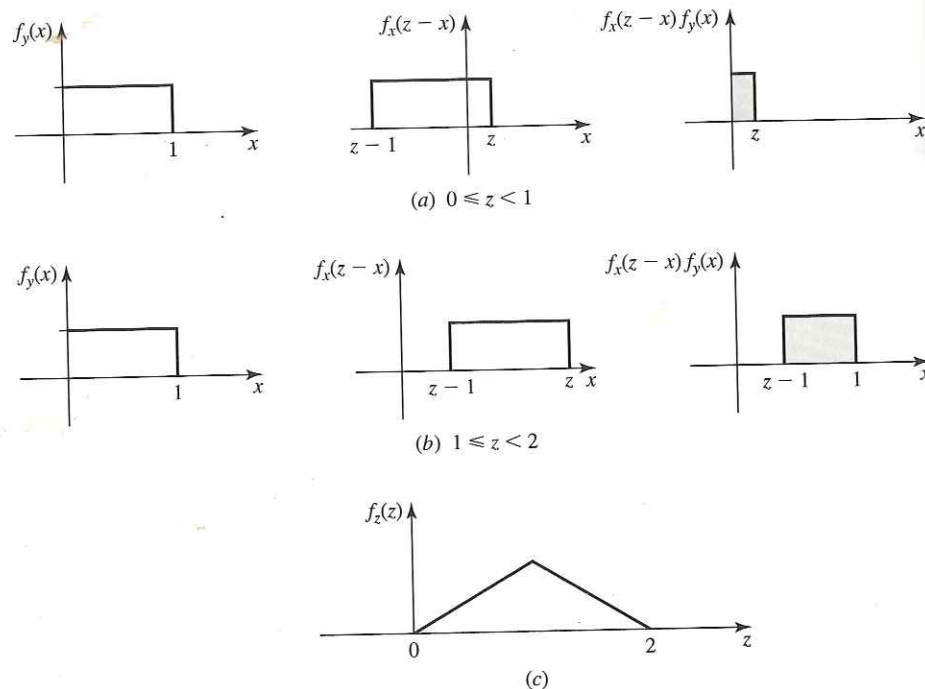$$f_z(z) = \int f_x(z-x) f_y(x) \, dx = \int_0^z 1 \, dx = z \qquad (6\text{-}51)$$

(a) $0 \le z < 1$

(b) $1 \le z < 2$

(c)

FIGURE 6-12

and for $1 \le z < 2$ (Fig. 6-12b)

$$f_z(z) = \int_{z-1}^{1} 1 \, dx = 2 - z \qquad (6\text{-}52)$$

Fig. 6-12c shows $f_z(z)$, which agrees with the convolution of two rectangular waveforms as well. ◀

EXAMPLE 6-9

$\mathbf{z} = \mathbf{x} - \mathbf{y}$

▶ Let $\mathbf{z} = \mathbf{x} - \mathbf{y}$. Determine $f_z(z)$.
From (6-37) and Fig. 6-13

$$F_z(z) = P\{\mathbf{x} - \mathbf{y} \le z\} = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{z+y} f_{xy}(x, y) \, dx \, dy$$

and hence

$$f_z(z) = \frac{dF_z(z)}{dz} = \int_{-\infty}^{\infty} f_{xy}(z + y, y) \, dy \qquad (6\text{-}53)$$

If $\mathbf{x}$ and $\mathbf{y}$ are independent, then this formula reduces to

$$f_z(z) = \int_{-\infty}^{\infty} f_x(z + y) f_y(y) \, dy = f_x(-z) \otimes f_y(y) \qquad (6\text{-}54)$$

which represents the convolution of $f_x(-z)$ with $f_y(z)$.
As a special case, suppose

$$f_x(x) = 0 \qquad x < 0, \qquad f_y(y) = 0 \qquad y < 0$$

In this case, $\mathbf{z}$ can be negative as well as positive, and that gives rise to two situations that should be analyzed separately, since the regions of integration for $z \ge 0$ and $z < 0$ are quite different.
For $z \ge 0$, from Fig. 6-14a

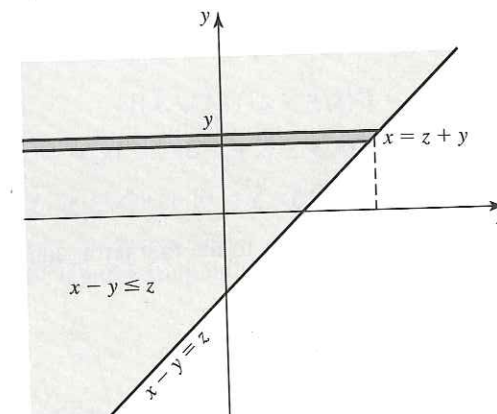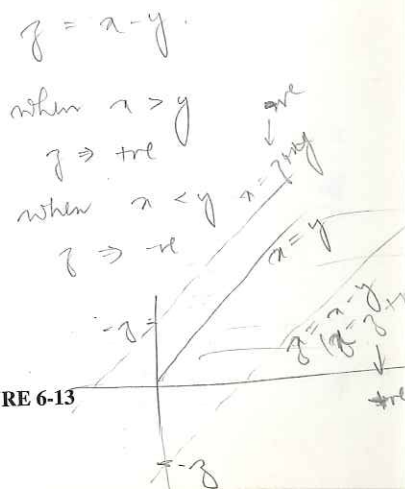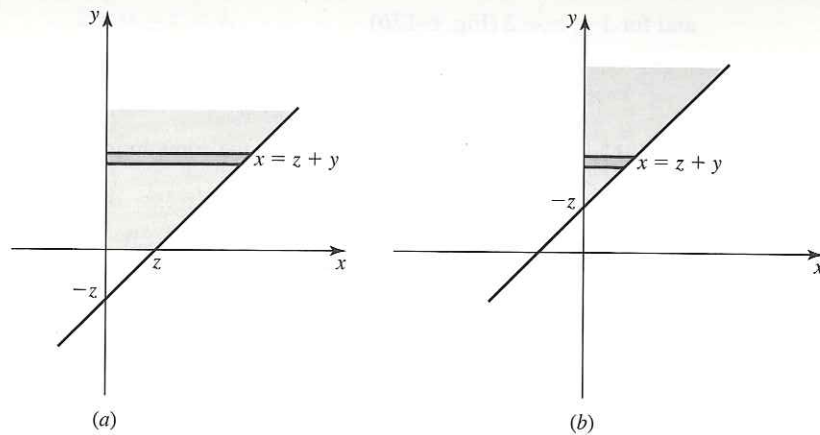$$F_z(z) = \int_{y=0}^{\infty} \int_{x=0}^{z+y} f_{xy}(x, y) \, dx \, dy$$

FIGURE 6-13

(a)   (b)

FIGURE 6-14

and for $z < 0$, from Fig. 6-14b

$$F_z(z) = \int_{y=-z}^{\infty} \int_{x=0}^{z+y} f_{xy}(x, y)\, dx\, dy$$

After differentiation, this gives

$$f_z(z) = \begin{cases} \int_0^{\infty} f_{xy}(z+y, y)\, dy & z \geq 0 \\ \int_{-z}^{\infty} f_{xy}(z+y, y)\, dy & z < 0 \end{cases} \tag{6-55}$$

◄

EXAMPLE 6-10

$\mathbf{z} = \mathbf{x}/\mathbf{y}$

► Let $\mathbf{z} = \mathbf{x}/\mathbf{y}$. Determine $f_z(z)$.
We have

$$F_z(z) = P\{\mathbf{x}/\mathbf{y} \leq z\} \tag{6-56}$$

The inequality $x/y \leq z$ can be rewritten as $\mathbf{x} \leq \mathbf{y}z$ if $\mathbf{y} > 0$, and $\mathbf{x} \geq \mathbf{y}z$ if $\mathbf{y} < 0$. Hence the event $\{\mathbf{x}/\mathbf{y} \leq z\}$ in (6-56) needs to be conditioned by the event $A = \{\mathbf{y} > 0\}$ and its compliment $\overline{A}$. Since $A \cup \overline{A} = S$, by the partition theorem, we have

$$P\{\mathbf{x}/\mathbf{y} \leq z\} = P\{\mathbf{x}/\mathbf{y} \leq z \cap (A \cup \overline{A})\}$$

$$= P\{\mathbf{x}/\mathbf{y} \leq z, \mathbf{y} > 0\} + P\{\mathbf{x}/\mathbf{y} \leq z, \mathbf{y} < 0\}$$

$$= P\{\mathbf{x} \leq \mathbf{y}z, \mathbf{y} > 0\} + P\{\mathbf{x} \geq \mathbf{y}z, \mathbf{y} < 0\} \tag{6-57}$$

Fig. 6-15a shows the area corresponding to the first term, and Fig. 6-15b shows that corresponding to the second term in (6-57).
Integrating over these two regions, we get

$$F_z(z) = \int_{y=0}^{\infty} \int_{x=-\infty}^{yz} f_{xy}(x, y)\, dx\, dy + \int_{y=-\infty}^{0} \int_{x=yz}^{\infty} f_{xy}(x, y)\, dx\, dy \tag{6-58}$$
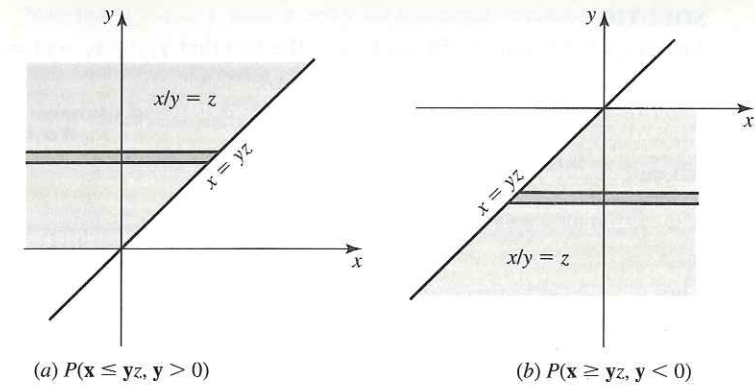

(a) $P(\mathbf{x} \leq \mathbf{y}z, \mathbf{y} > 0)$ (b) $P(\mathbf{x} \geq \mathbf{y}z, \mathbf{y} < 0)$

FIGURE 6-15

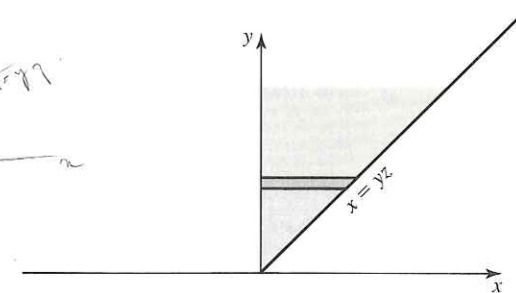**FIGURE 6-16**

Differentiation gives

$$f_z(z) = \int_0^{\infty} y f_{xy}(yz, y)\, dy + \int_{-\infty}^0 -y f_{xy}(yz, y)\, dy$$

$$= \int_{-\infty}^{\infty} |y| f_{xy}(yz, y)\, dy \tag{6-59}$$

Note that if $\mathbf{x}$ and $\mathbf{y}$ are non-negative random variables, then the area of integration reduces to that shown in Fig. 6-16.
This gives

$$F_z(z) = \int_{y=0}^{\infty} \int_{x=0}^{yz} f_{xy}(x, y)\, dx\, dy$$

or

$$f_z(z) = \int_{y=0}^{\infty} y f_{xy}(yz, y)\, dy \tag{6-60}$$

◄

EXAMPLE 6-11

► $\mathbf{x}$ and $\mathbf{y}$ are jointly normal random variables with zero mean and

$$f_{xy}(x, y) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1 - r^2}} e^{-\left[\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)\right]} \tag{6-61}$$

Show that the ratio $\mathbf{z} = \mathbf{x}/\mathbf{y}$ has a Cauchy density centered at $r\sigma_1/\sigma_2$.

SOLUTION

Inserting (6-61) into (6-59) and using the fact that $f_{xy}(-x, -y) = f_{xy}(x, y)$, we obtain

$$f_z(z) = \frac{2}{2\pi \sigma_1 \sigma_2 \sqrt{1 - r^2}} \int_0^\infty y e^{-y^2/2\sigma_0^2} \, dy = \frac{\sigma_0^2}{\pi \sigma_1 \sigma_2 \sqrt{1 - r^2}}$$

where

$$\sigma_0^2 = \frac{1 - r^2}{(z^2/\sigma_1^2) - (2rz/\sigma_1\sigma_2) + (1/\sigma_2^2)}$$

Thus

$$f_z(z) = \frac{\sigma_1 \sigma_2 \sqrt{1 - r^2}/\pi}{\sigma_2^2(z - r\sigma_1/\sigma_2)^2 + \sigma_1^2(1 - r^2)} \qquad (6\text{-}62)$$

which represents a Cauchy random variable centered at $r\sigma_1/\sigma_2$. Integrating (6-62) from $-\infty$ to z, we obtain the corresponding distribution function to be

$$F_z(z) = \frac{1}{2} + \frac{1}{\pi} \arctan \frac{\sigma_2 z - r\sigma_1}{\sigma_1 \sqrt{1 - r^2}} \qquad (6\text{-}63)$$

◀

As an application, we can use (6-63) to determine the probability masses m_1, m_2, m_3, and m_4 in the four quadrants of the xy plane for (6-61). From the spherical symmetry of (6-61), we have

$$m_1 = m_3 \qquad m_2 = m_4$$

But the second and fourth quadrants represent the region of the plane where $x/y < 0$. The probability that the point (x, y) is in that region equals, therefore, the probability that the random variable $\mathbf{z} = \mathbf{x/y}$ is negative. Thus

$$m_2 + m_4 = P(\mathbf{z} \le 0) = F_z(0) = \frac{1}{2} - \frac{1}{\pi} \arctan \frac{r}{\sqrt{1 - r^2}}$$

and

$$m_1 + m_3 = 1 - (m_2 + m_4) = \frac{1}{2} + \frac{1}{\pi} \arctan \frac{r}{\sqrt{1 - r^2}}$$

If we define $\alpha = \arctan r/\sqrt{1 - r^2}$, this gives

$$m_1 = m_3 = \frac{1}{4} + \frac{\alpha}{2\pi} \qquad m_2 = m_4 = \frac{1}{4} - \frac{\alpha}{2\pi} \qquad (6\text{-}64)$$

Of course, we could have obtained this result by direct integration of (6-61) in each quadrant. However, this is simpler.

EXAMPLE 6-12

▶ Let $\mathbf{x}$ and $\mathbf{y}$ be independent gamma random variables with $\mathbf{x} \sim G(m, \alpha)$ and $\mathbf{y} \sim G(n, \alpha)$. Show that $\mathbf{z} = \mathbf{x/(x + y)}$ has a beta distribution.

Proof. $\qquad f_{xy}(x, y) = f_x(x) f_y(y)$

$$= \frac{1}{\alpha^{m+n} \Gamma(m) \Gamma(n)} x^{m-1} y^{n-1} e^{-(x+y)/\alpha} \qquad x > 0 \qquad y > 0 \qquad (6\text{-}65)$$

Note that $0 < z < 1$, since $\mathbf{x}$ and $\mathbf{y}$ are non-negative random variables

$$F_z(z) = P\{z \le z\} = P\left(\frac{\mathbf{x}}{\mathbf{x} + \mathbf{y}} \le z\right) = P\left(\mathbf{x} \le \mathbf{y}\frac{z}{1 - z}\right)$$

$$= \int_0^\infty \int_0^{yz/(1-z)} f_{xy}(x, y) \, dx \, dy$$

where we have made use of Fig. 6-16. Differentiation with respect to z gives

$$f_z(z) = \int_0^\infty \frac{y}{(1 - z)^2} f_{xy}(yz/(1 - z), y) \, dy$$

$$= \int_0^\infty \frac{y}{(1 - z)^2} \frac{1}{\alpha^{m+n} \Gamma(m) \Gamma(n)} \left(\frac{yz}{1 - z}\right)^{m-1} y^{n-1} e^{-y/(1-z)\alpha} \, dy$$

$$= \frac{1}{\alpha^{m+n} \Gamma(m) \Gamma(n)} \frac{z^{m-1}}{(1 - z)^{m+1}} \int_0^\infty y^{m+n-1} e^{-y/\alpha(1-z)} \, dy$$

$$= \frac{z^{m-1}(1 - z)^{n-1}}{\Gamma(m) \Gamma(n)} \int_0^\infty u^{m+n-1} e^{-u} \, du = \frac{\Gamma(m + n)}{\Gamma(m) \Gamma(n)} z^{m-1}(1 - z)^{n-1}$$

$$= \begin{cases} \dfrac{1}{\beta(m, n)} z^{m-1}(1 - z)^{n-1} & 0 < z < 1 \\ 0 & \text{otherwise} \end{cases} \qquad (6\text{-}66)$$

which represents a beta distribution. ◀

EXAMPLE 6-13

$\mathbf{z} = \mathbf{x}^2 + \mathbf{y}^2$

▶ Let $\mathbf{z} = \mathbf{x}^2 + \mathbf{y}^2$. Determine $f_z(z)$.

We have

$$F_z(z) = P\{x^2 + y^2 \le z\} = \iint_{x^2 + y^2 \le z} f_{xy}(x, y) \, dx \, dy$$

But, $\mathbf{x}^2 + \mathbf{y}^2 \le z$ represents the area of a circle with radius $\sqrt{z}$, and hence (see Fig. 6-17)

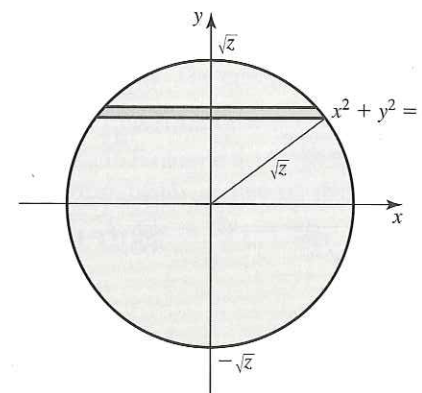$$F_z(z) = \int_{y=-\sqrt{z}}^{\sqrt{z}} \int_{x=-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f_{xy}(x, y) \, dx \, dy$$

FIGURE 6-17

This gives

$$f_z(z) = \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{2\sqrt{z-y^2}} \{f_{xy}(\sqrt{z-y^2}, y) + f_{xy}(-\sqrt{z-y^2}, y)\} \, dy \quad (6\text{-}67)$$

◀

As an illustration, consider Example 6-14.

EXAMPLE 6-14 ▶ **x** and **y** are independent normal random variables with zero mean and common variance σ^2. Determine $f_z(z)$ for $\mathbf{z} = \mathbf{x}^2 + \mathbf{y}^2$.

SOLUTION
Using (6-67), we get

$$f_z(z) = \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{2\sqrt{z-y^2}} \left(2 \cdot \frac{1}{2\pi\sigma^2} e^{(z-y^2+y^2)/2\sigma^2} \right) dy$$

$$= \frac{e^{-z/2\sigma^2}}{\pi\sigma^2} \int_0^{\sqrt{z}} \frac{1}{\sqrt{z-y^2}} \, dy = \frac{e^{-z/2\sigma^2}}{\pi\sigma^2} \int_0^{\pi/2} \frac{\sqrt{z}\cos\theta}{\sqrt{z}\cos\theta} \, d\theta$$

$$= \frac{1}{2\sigma^2} e^{-z/2\sigma^2} U(z) \quad (6\text{-}68)$$

where we have used the substitution $y = \sqrt{z}\sin\theta$. From (6-68), we have the following: If **x** and **y** are independent zero mean Gaussian random variables with common variance σ^2, then $\mathbf{x}^2 + \mathbf{y}^2$ is an exponential random variable with parameter $2\sigma^2$. ◀

EXAMPLE 6-15

$\mathbf{z} = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$

▶ Let $\mathbf{z} = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$. Find $f_z(z)$.

SOLUTION
From Fig. 6-17, the present case corresponds to a circle with radius z^2. Thus

$$F_z(z) = \int_{y=-z}^{z} \int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f_{xy}(x, y) \, dx \, dy$$

and by differentiation,

$$f_z(z) = \int_{-z}^{z} \frac{z}{\sqrt{z^2-y^2}} \{f_{xy}(\sqrt{z^2-y^2}, y) + f_{xy}(-\sqrt{z^2-y^2}, y)\} \, dy \quad (6\text{-}69)$$

In particular, if **x** and **y** are zero mean independent Gaussian random variables as in the previous example, then

$$f_z(z) = 2 \int_0^z \frac{z}{\sqrt{z^2-y^2}} \frac{2}{2\pi\sigma^2} e^{-(z^2-y^2+y^2)/2\sigma^2} \, dy$$

$$= \frac{2z}{\pi\sigma^2} e^{-z^2/2\sigma^2} \int_0^z \frac{1}{\sqrt{z^2-y^2}} \, dy = \frac{2z}{\pi\sigma^2} e^{-z^2/2\sigma^2} \int_0^{\pi/2} \frac{z\cos\theta}{z\cos\theta} \, d\theta$$

$$= \frac{z}{\sigma^2} e^{-z^2/2\sigma^2} U(z) \quad (6\text{-}70)$$

which represents a Rayleigh distribution. Thus, if $\mathbf{w} = \mathbf{x} + i\mathbf{y}$, where **x** and **y** are real independent normal random variables with zero mean and equal variance, then

the random variable $|\mathbf{w}| = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$ has a Rayleigh density. **w** is said to be a complex Gaussian random variable with zero mean, if its real and imaginary parts are independent. So far we have seen that the magnitude of a complex Gaussian random variable has Rayleigh distribution. What about its phase

$$\theta = \tan^{-1}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) \quad (6\text{-}71)$$

Clearly, the principal value of θ lies in the interval $(-\pi/2, \pi/2)$. If we let $\mathbf{u} = \tan\theta = \mathbf{y}/\mathbf{x}$, then from Example 6-11, **u** has a Cauchy distribution (see (6-62) with $\sigma_1 = \sigma_2$, $r = 0$)

$$f_u(u) = \frac{1/\pi}{u^2 + 1} \qquad -\infty < u < \infty$$

As a result, the principal value of θ has the density function

$$f_\theta(\theta) = \frac{1}{|d\theta/du|} f_u(\tan\theta) = \frac{1}{(1/\sec^2\theta)} \frac{1/\pi}{\tan^2\theta + 1}$$

$$= \begin{cases} 1/\pi & -\pi/2 < \theta < \pi/2 \\ 0 & \text{otherwise} \end{cases} \quad (6\text{-}72)$$

However, in the representation $\mathbf{x} + j\mathbf{y} = \mathbf{r}e^{j\theta}$, the variable θ lies in the interval $(-\pi, \pi)$, and taking into account this scaling by a factor of two, we obtain

$$f_\theta(\theta) = \begin{cases} 1/2\pi & -\pi < \theta < \pi \\ 0 & \text{otherwise} \end{cases} \quad (6\text{-}73)$$

To summarize, the magnitude and phase of a zero mean complex Gaussian random variable have Rayleigh and uniform distributions respectively. Interestingly, as we will show later (Example 6-22), these two derived random variables are also statistically *independent* of each other! ◀

Let us reconsider Example 6-15 where **x** and **y** are independent Gaussian random variables with nonzero means μ_x and μ_y respectively. Then $\mathbf{z} = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$ is said to be a Rician random variable. Such a scene arises in fading multipath situations where there is a dominant constant component (mean) in addition to a zero mean Gaussian random variable. The constant component may be the line of sight signal and the zero mean Gaussian random variable part could be due to random multipath components adding up incoherently. The envelope of such a signal is said to be Rician instead of Rayleigh.

EXAMPLE 6-16 ▶ Redo Example 6-15, where **x** and **y** are independent Gaussian random variables with nonzero means μ_x and μ_y respectively.

SOLUTION
Since

$$f_{xy}(x, y) = \frac{1}{2\pi\sigma^2} e^{-[(x-\mu_x)^2+(y-\mu_y)^2]/2\sigma^2}$$

substituting this into (6-69) and letting $y = z\sin\theta$, $\mu = \sqrt{\mu_x^2 + \mu_y^2}$, $\mu_x = \mu\cos\phi$,

$\mu_y = \mu \sin \phi$, we get the Rician distribution to be

$$f_z(z) = \frac{ze^{-(z^2+\mu^2)/2\sigma^2}}{2\pi\sigma^2} \int_{-\pi/2}^{\pi/2} \left(e^{z\mu\cos(\theta-\phi)/\sigma^2} + e^{-z\mu\cos(\theta+\phi)/\sigma^2} \right) d\theta$$

$$= \frac{ze^{-(z^2+\mu^2)/2\sigma^2}}{2\pi\sigma^2} \left(\int_{-\pi/2}^{\pi/2} e^{z\mu\cos(\theta-\phi)/\sigma^2} \, d\theta + \int_{\pi/2}^{3\pi/2} e^{z\mu\cos(\theta-\phi)/\sigma^2} \, d\theta \right)$$

$$= \frac{ze^{-(z^2+\mu^2)/2\sigma^2}}{\sigma^2} I_0 \left(\frac{z\mu}{\sigma^2} \right) \tag{6-74}$$

where

$$I_0(\eta) \triangleq \frac{1}{2\pi} \int_0^{2\pi} e^{\eta\cos(\theta-\phi)} \, d\theta = \frac{1}{\pi} \int_0^{\pi} e^{\eta\cos\theta} \, d\theta$$

is the modified Bessel function of the first kind and zeroth order. ◀

Order Statistics

In general, given any n-tuple $x_1, x_2, \ldots, x_n$, we can rearrange them in an increasing order of magnitude such that

$$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$$

where $x_{(1)} = \min(x_1, x_2, \ldots, x_n)$, and $x_{(2)}$ is the second smallest value among $x_1, x_2, \ldots, x_n$, and finally $x_{(n)} = \max(x_1, x_2, \ldots, x_n)$. The functions *min* and *max* are nonlinear operators, and represent special cases of the more general order statistics. If $x_1, x_2, \ldots, x_n$ represent random variables, the function $x_{(k)}$ that takes on the value $x_{(k)}$ in each possible sequence $(x_1, x_2, \ldots, x_n)$ is known as the kth-order statistic. $\{x_{(1)}, x_{(2)}, \ldots, x_{(n)}\}$ represent the set of order statistics among n random variables. In this context

$$R = x_{(n)} - x_{(1)} \tag{6-75}$$

represents the range, and when $n = 2$, we have the *max* and *min* statistics.

Order statistics is useful when relative magnitude of observations is of importance. When worst case scenarios have to be accounted for, then the function $max(\cdot)$ is quite useful. For example, let $x_1, x_2, \ldots, x_n$ represent the recorded flood levels over the past n years at some location. If the objective is to construct a dam to prevent any more flooding, then the height H of the proposed dam should satisfy the inequality

$$H > \max(x_1, x_2, \ldots, x_n) \tag{6-76}$$

with some finite probability. In that case, the p.d.f. of the random variable on the right side of (6-76) can be used to compute the desired height. In another case, if a bulb manufacturer wants to determine the average time to failure (μ) of its bulbs based on a sample of size n, the sample mean $(x_1 + x_2 + \cdots + x_n)/n$ can be used as an estimate for μ. On the other hand, an estimate based on the least time to failure has other attractive features. This estimate $\min(x_1, x_2, \ldots, x_n)$ may not be as good as the sample mean in terms of their respective variances, but the $min(\cdot)$ can be computed as soon as the first bulb fuses, whereas to compute the sample mean one needs to wait till the last of the lot extinguishes.

EXAMPLE 6-17

$z = \max(x, y)$
$w = \min(x, y)$

▶ Let $z = \max(x, y)$ and $w = \min(x, y)$. Determine $f_z(z)$ and $f_w(w)$.

$$z = \max(x, y) = \begin{cases} x & x > y \\ y & x \leq y \end{cases} \tag{6-77}$$

we have [see (6-57)]

$$\begin{aligned} F_z(z) &= P\{\max(x, y) \leq z\} \\ &= P\{(x \leq z, x > y) \cup (y \leq z, x \leq y)\} \\ &= P\{x \leq z, x > y\} + P\{y \leq z, x \leq y\} \end{aligned}$$

since $\{x > y\}$ and $\{x \leq y\}$ are mutually exclusive sets that form a partition. Figure 6-18a and 6-18b show the regions satisfying the corresponding inequalities in each term seen here.

Figure 6-18c represents the total region, and from there

$$F_z(z) = P\{x \leq z, y \leq z\} = F_{xy}(z, z) \tag{6-78}$$

If x and y are independent, then

$$F_z(z) = F_x(z)F_y(z)$$

(a) $P\{x \leq z, x > y\}$

(b) $P\{y \leq z, x \leq y\}$

(c)

FIGURE 6-18

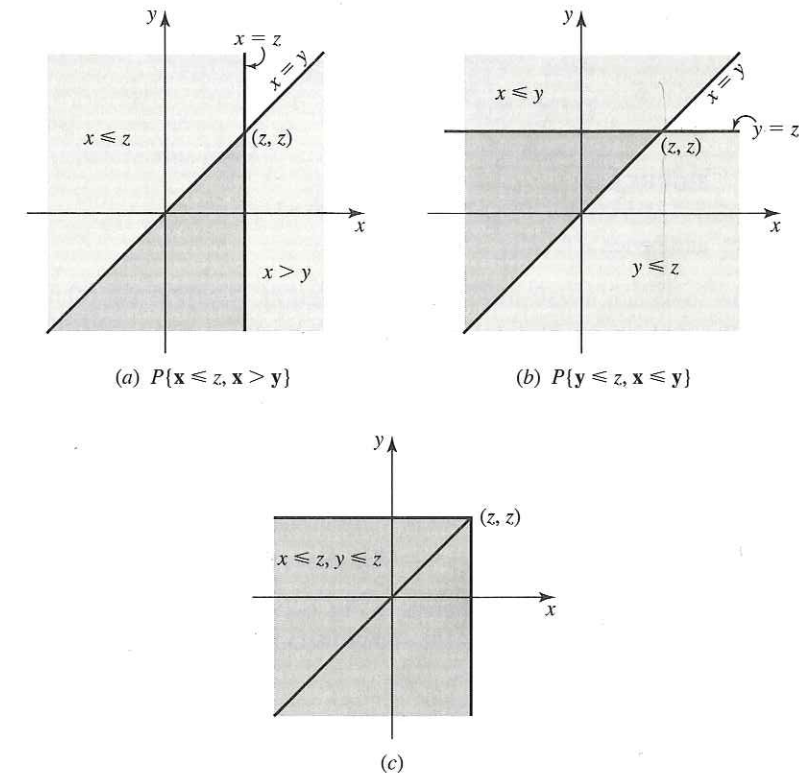(a) $P\{y \le w, x > y\}$
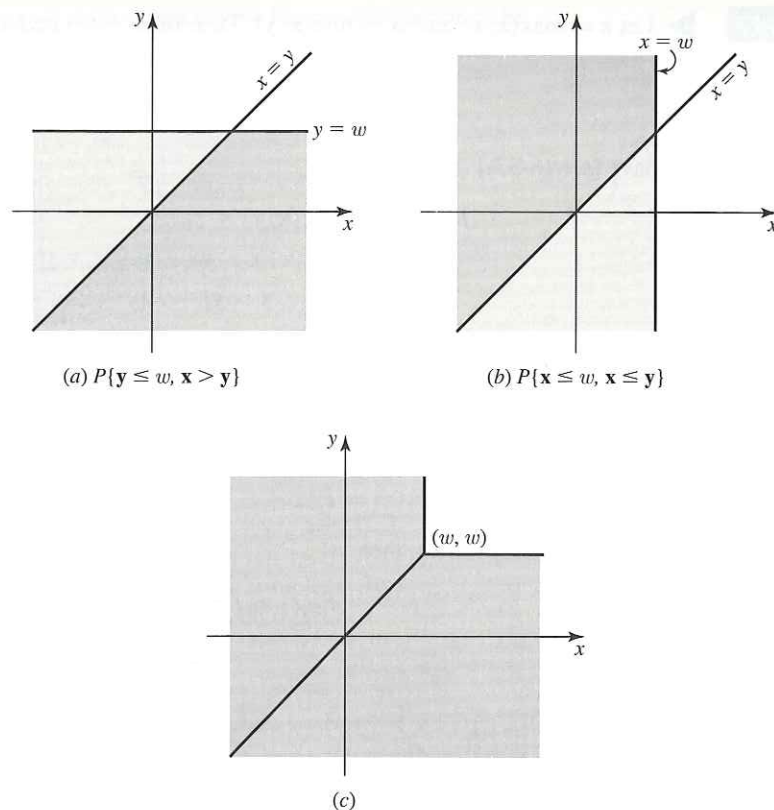

(b) $P\{x \le w, x \le y\}$

(c)

FIGURE 6-19

and hence

$$f_z(z) = F_x(z)f_y(z) + f_x(z)F_y(z) \qquad (6\text{-}79)$$

Similarly,

$$w = \min(x, y) = \begin{cases} y & x > y \\ x & x \le y \end{cases} \qquad (6\text{-}80)$$

Thus,

$$\begin{aligned} F_w(w) &= P\{\min(x, y) \le w\} \\ &= P\{y \le w, x > y\} + P\{x \le w, x \le y\} \end{aligned}$$

Once again, the shaded areas in Fig. 6-19a and 6-19b show the regions satisfying these inequalities, and Fig. 6-19c shows them together.

From Fig. 6-19c,

$$\begin{aligned} F_w(w) &= 1 - P\{w > w\} = 1 - P\{x > w, y > w\} \\ &= F_x(w) + F_y(w) - F_{xy}(w, w) \end{aligned} \qquad (6\text{-}81)$$

where we have made use of (6-4) with $x_2 = y_2 = \infty$, and $x_1 = y_1 = w$. ◀

EXAMPLE 6-18 ▶ Let x and y be independent exponential random variables with common parameter λ. Define $w = \min(x, y)$. Find $f_w(w)$.

SOLUTION
From (6-81)

$$F_w(w) = F_x(w) + F_y(w) - F_x(w)F_y(w)$$

and hence

$$f_w(w) = f_x(w) + f_y(w) - f_x(w)F_y(w) - F_x(w)f_y(w)$$

But $f_x(w) = f_y(w) = \lambda e^{-\lambda w}$, and $F_x(w) = F_y(w) = 1 - e^{-\lambda w}$, so that

$$f_w(w) = 2\lambda e^{\lambda w} - 2(1 - e^{-\lambda w})\lambda e^{-\lambda w} = 2\lambda e^{-2\lambda w}U(w) \qquad (6\text{-}82)$$

Thus $\min(x, y)$ is also exponential with parameter 2λ. ◀

EXAMPLE 6-19 ▶ Suppose x and y are as given in Example 6-18. Define

$$z = \frac{\min(x, y)}{\max(x, y)}$$

Although $\min(\cdot)/\max(\cdot)$ represents a complicated function, by partitioning the whole space as before, it is possible to simplify this function. In fact

$$z = \begin{cases} x/y & x \le y \\ y/x & x > y \end{cases} \qquad (6\text{-}83)$$

As before, this gives

$$\begin{aligned} F_z(z) &= P\{x/y \le z, x \le y\} + P\{y/x \le z, x > y\} \\ &= P\{x \le yz, x \le y\} + P\{y \le xz, x > y\} \end{aligned}$$

Since x and y are both positive random variables in this case, we have $0 < z < 1$. The shaded regions in Fig. 6-20a and 6-20b represent the two terms in this sum.
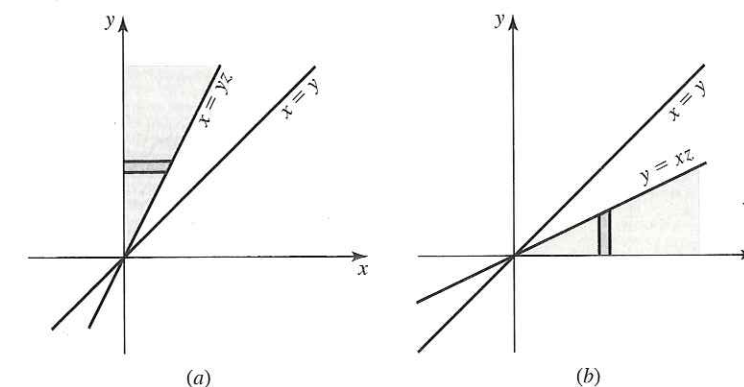

(a)

(b)

FIGURE 6-20

From Fig. 6-20,

$$F_z(z) = \int_0^\infty \int_{x=0}^{yz} f_{xy}(x, y)\,dx\,dy + \int_0^\infty \int_{y=0}^{xz} f_{xy}(x, y)\,dy\,dx$$

Hence

$$f_z(z) = \int_0^\infty y f_{xy}(yz, y)\,dy + \int_0^\infty x f_{xy}(x, xz)\,dx$$

$$= \int_0^\infty y\big(f_{xy}(yz, y) + f_{xy}(y, yz)\big)\,dy$$

$$= \int_0^\infty y\lambda^2\big(e^{-\lambda(yz+y)} + e^{-\lambda(y+yz)}\big)\,dy$$

$$= 2\lambda^2 \int_0^\infty y e^{-\lambda(1+z)y}\,dy = \frac{2}{(1+z)^2}\int_0^\infty u e^{-u}\,du$$

$$= \begin{cases} \dfrac{2}{(1+z)^2} & 0 < z < 1 \\ 0 & \text{otherwise} \end{cases} \tag{6-84}$$

◀

EXAMPLE 6-20

DISCRETE CASE

▶ Let x and y be independent Poisson random variables with parameters λ_1 and λ_2, respectively. Let $z = x + y$. Determine the p.m.f. of z.

Since x and y both take values $\{0, 1, 2, \ldots\}$, the same is true for z. For any $n = 0, 1, 2, \ldots$, $\{x+y = n\}$ gives only a finite number of options for x and y. In fact, if $x = 0$, then y must be n; if $x = 1$, then y must be $n - 1$, and so on. Thus the event $\{x + y = n\}$ is the union of mutually exclusive events $A_k = \{x = k, y = n - k\}, k = 0 \to n$.

$$P\{z = n\} = P\{x + y = n\} = P\left(\bigcup_{k=0}^n \{x = k, y = n - k\}\right)$$

$$= \sum_{k=0}^n P\{x = k, y = n - k\} \tag{6-85}$$

If x and y are also independent, then

$$P\{x = k, y = n - k\} = P\{x = k\}P\{y = n - k\}$$

and hence

$$P\{z = n\} = \sum_{k=0}^n P\{x = k\}P\{y = n - k\}$$

$$= \sum_{k=0}^n e^{-\lambda_1}\frac{\lambda_1^k}{k!} e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n - k)!} = \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^n \frac{n!}{k!(n - k)!}\lambda_1^k\lambda_2^{n-k}$$

$$= e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1 + \lambda_2)^n}{n!}, \quad n = 0, 1, 2, \ldots, \infty \tag{6-86}$$

Thus z represents a Poisson random variable with parameter $\lambda_1 + \lambda_2$, indicating that sum of independent Poisson random variables is a Poisson random variable whose parameter is the sum of the parameters of the original random variables.

As Example 6-20 indicates, this procedure is too tedious in the discrete case. As we shall see in Sec. 6-5, the joint characteristic function or the moment generating function can be used to solve problems of this type in a much easier manner. ◀

6-3 TWO FUNCTIONS OF TWO RANDOM VARIABLES

In the spirit of the previous section, let us look at an immediate generalization. Suppose x and y are two random variables with joint p.d.f. $f_{xy}(x, y)$. Given two functions $g(x, y)$ and $h(x, y)$, define two new random variables

$$z = g(x, y) \tag{6-87}$$

$$w = h(x, y) \tag{6-88}$$

How does one determine their joint p.d.f. $f_{zw}(z, w)$? Obviously with $f_{zw}(z, w)$ in hand, the marginal p.d.f.s $f_z(z)$ and $f_w(w)$ can be easily determined.

The procedure for determining $f_{zw}(z, w)$ is the same as that in (6-36). In fact for given numbers z and w,

$$F_{zw}(z, w) = P\{z(\xi) \le z, w(\xi) \le w\} = P\{g(x, y) \le z, h(x, y) \le w\}$$

$$= P\{(x, y) \in D_{z,w}\} = \iint\limits_{(x,y)\in D_{z,w}} f_{xy}(x, y)\,dx\,dy \tag{6-89}$$

where $D_{z,w}$ is the region in the xy plane such that the inequalities $g(x, y) \le z$ and $h(x, y) \le w$ are *simultaneously* satisfied in Fig. 6-21.

We illustrate this technique in Example 6-21.

EXAMPLE 6-21

▶ Suppose x and y are independent uniformly distributed random variables in the interval $(0, \theta)$. Define $z = \min(x, y), w = \max(x, y)$. Determine $f_{zw}(z, w)$.

SOLUTION

Obviously both z and w vary in the interval $(0, \theta)$. Thus

$$F_{zw}(z, w) = 0 \quad \text{if} \quad z < 0 \quad \text{or} \quad w < 0 \tag{6-90}$$

$$F_{zw}(z, w) = P\{z \le z, w \le w\} = P\{\min(x, y) \le z, \max(x, y) \le w\} \tag{6-91}$$

$$= P\{x \le z, y \le w\}$$
$$\quad :x \le y$$
$$\to P\left\{y \le z, x \le w \atop \text{when } x > y\right\}$$
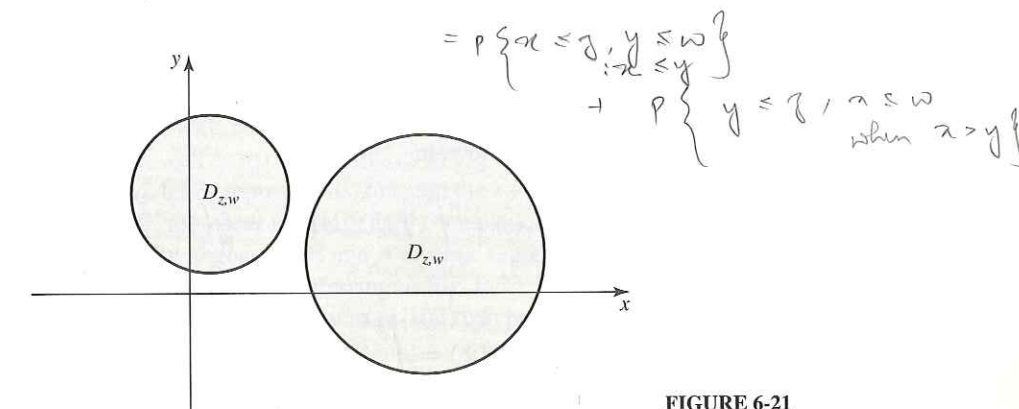
FIGURE 6-21
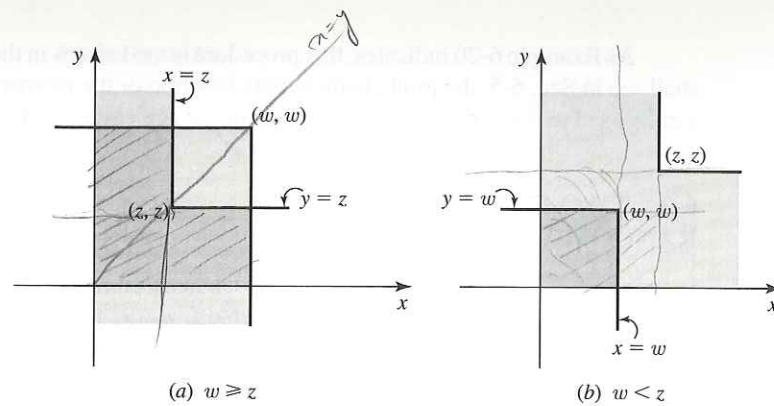
(a) $w \geq z$ (b) $w < z$

FIGURE 6-22

We must consider two cases: $w \geq z$ and $w < z$, since they give rise to different regions for $D_{z,w}$ (see Fig. 6-22a and 6-22b).

For $w \geq z$, from Fig. 6.22a, the region $D_{z,w}$ is represented by the doubly shaded area (see also Fig. 6-18c and Fig. 6-19c). Thus

$$F_{zw}(z, w) = F_{xy}(z, w) + F_{xy}(w, z) - F_{xy}(z, z) \qquad w \geq z \qquad (6\text{-}92)$$

and for $w < z$, from Fig. 6.22b, we obtain

$$F_{zw}(z, w) = F_{xy}(w, w) \qquad w < z \qquad (6\text{-}93)$$

with

$$F_{xy}(x, y) = F_x(x) F_y(y) = \frac{x}{\theta} \cdot \frac{y}{\theta} = \frac{xy}{\theta^2} \qquad (6\text{-}94)$$

we obtain

$$F_{zw}(z, w) = \begin{cases} (2wz - z^2)/\theta^2 & 0 < z < w < \theta \\ w^2/\theta^2 & 0 < w < z < \theta \end{cases} \qquad (6\text{-}95)$$

Thus

$$f_{zw}(z, w) = \begin{cases} 2/\theta^2 & 0 < z < w < \theta \\ 0 & \text{otherwise} \end{cases} \qquad (6\text{-}96)$$

From (6-96), we also obtain

$$f_z(z) = \int_z^\theta f_{zw}(z, w)\, dw = \frac{2}{\theta}\left(1 - \frac{z}{\theta}\right) \qquad 0 < z < \theta \qquad (6\text{-}97)$$

and

$$f_w(w) = \int_0^w f_{zw}(z, w)\, dz = \frac{2w}{\theta^2} \qquad 0 < w < \theta \qquad (6\text{-}98)$$
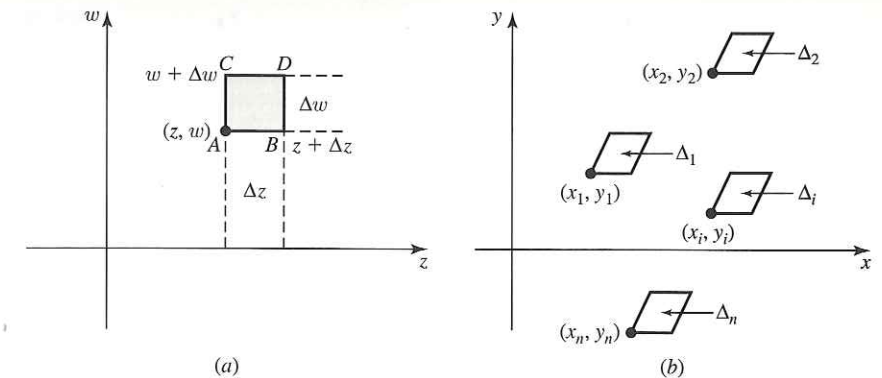
◀

(a)   (b)

FIGURE 6-23

Joint Density

If $g(x, y)$ and $h(x, y)$ are continuous and differentiable functions, then, as in the case of one random variable [see (5-16)], it is possible to develop a formula to obtain the joint p.d.f. $f_{zw}(z, w)$ directly. Toward this, consider the equations

$$g(x, y) = z \qquad h(x, y) = w \qquad (6\text{-}99)$$

For a given point (z, w), equation (6-99) can have many solutions. Let us say (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$ represent these multiple solutions such that (see Fig. 6-23)

$$g(x_i, y_i) = z \qquad h(x_i, y_i) = w \qquad (6\text{-}100)$$

Consider the problem of evaluating the probability

$$P\{z < \mathbf{z} \leq z + \Delta z, w < \mathbf{w} \leq w + \Delta w\}$$
$$= P\{z < g(\mathbf{x}, \mathbf{y}) \leq z + \Delta z, w < h(\mathbf{x}, \mathbf{y}) \leq w + \Delta w\} \qquad (6\text{-}101)$$

Using (6-8) we can rewrite (6-101) as

$$P\{z < \mathbf{z} \leq z + \Delta z, w < \mathbf{w} \leq w + \Delta w\} = f_{zw}(z, w)\, \Delta z\, \Delta w \qquad (6\text{-}102)$$

But to translate this probability in terms of $f_{xy}(x, y)$, we need to evaluate the equivalent region for $\Delta z\, \Delta w$ in the xy plane. Toward this, referring to Fig. 6-24, we observe that the point A with coordinates (z, w) gets mapped onto the point A' with coordinates (x_i, y_i) (as well as to other points as in Fig 6.23b). As z changes to $z + \Delta z$ to point B in Fig. 6.24a, let B' represent its image in the xy plane. Similarly, as w changes to $w + \Delta w$ to C, let C' represent its image in the xy plane.

Finally D goes to D', and $A'B'C'D'$ represents the equivalent parallelogram in the xy plane with area Δ_i. Referring to Fig. 6-23, because of the nonoverlapping nature of these regions the probability in (6-102) can be alternatively expressed as

$$\sum_i P\{(\mathbf{x}, \mathbf{y}) \in \Delta_i\} = \sum_i f_{xy}(x_i, y_i)\, \Delta_i \qquad (6\text{-}103)$$
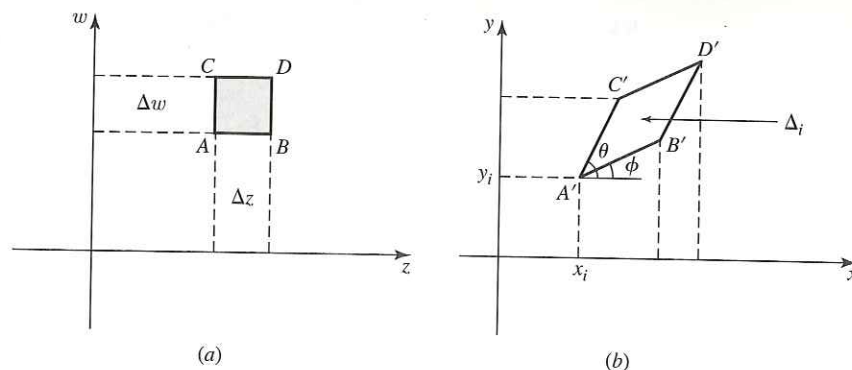
(a)   (b)

FIGURE 6-24

Equating (6-102) and (6-103) we obtain

$$f_{zw}(z, w) = \sum_i f_{xy}(x_i, y_i) \frac{\Delta_i}{\Delta z \, \Delta w} \tag{6-104}$$

To simplify (6-104), we need to evaluate the area Δ_i of the parallelograms in Fig. 6.24b in terms of $\Delta z \, \Delta w$. Toward this, let g_1 and h_1 denote the inverse transformation in (6-99), so that

$$x_i = g_1(z, w) \qquad y_i = h_1(z, w) \tag{6-105}$$

As the point (z, w) goes to $(x_i, y_i) \equiv A'$, the point $(z + \Delta z, w)$ goes to B', the point $(z, w + \Delta w)$ goes to C', and the point $(z + \Delta z, w + \Delta w)$ goes to D'. Hence the respective x and y coordinates of B' are given by

$$g_1(z + \Delta z, w) = g_1(z, w) + \frac{\partial g_1}{\partial z} \Delta z = x_i + \frac{\partial g_1}{\partial z} \Delta z \tag{6-106}$$

and

$$h_1(z + \Delta z, w) = h_1(z, w) + \frac{\partial h_1}{\partial z} \Delta z = y_i + \frac{\partial h_1}{\partial z} \Delta z \tag{6-107}$$

Similarly those of C' are given by

$$x_i + \frac{\partial g_1}{\partial w} \Delta w \qquad y_i + \frac{\partial h_1}{\partial w} \Delta w \tag{6-108}$$

The area of the parallelogram $A'B'C'D'$ in Fig. 6-24b is given by

$$\Delta_i = (A'B')(A'C') \sin(\theta - \phi)$$

$$= (A'B' \cos \phi)(A'C' \sin \theta) - (A'B' \sin \phi)(A'C' \cos \theta) \tag{6-109}$$

But from Fig. 6-24b, and (6-106)–(6-108).

$$A'B' \cos \phi = \frac{\partial g_1}{\partial z} \Delta z \qquad A'C' \sin \theta = \frac{\partial h_1}{\partial w} \Delta w \tag{6-110}$$

$$A'B' \sin \phi = \frac{\partial h_1}{\partial z} \Delta z \qquad A'C' \cos \theta = \frac{\partial g_1}{\partial w} \Delta w \tag{6-111}$$

so that

$$\Delta_i = \left(\frac{\partial g_1}{\partial z} \frac{\partial h_1}{\partial w} - \frac{\partial g_1}{\partial w} \frac{\partial h_1}{\partial z} \right) \Delta z \, \Delta w \tag{6-112}$$

and

$$\frac{\Delta_i}{\Delta z \, \Delta w} = \left(\frac{\partial g_1}{\partial z} \frac{\partial h_1}{\partial w} - \frac{\partial g_1}{\partial w} \frac{\partial h_1}{\partial z} \right) = \begin{vmatrix} \dfrac{\partial g_1}{\partial z} & \dfrac{\partial g_1}{\partial w} \\ \dfrac{\partial h_1}{\partial z} & \dfrac{\partial h_1}{\partial w} \end{vmatrix} \tag{6-113}$$

The determinant on the right side of (6-113) represents the absolute value of the Jacobian $J(z, w)$ of the inverse transformation in (6-105). Thus

$$J(z, w) = \begin{vmatrix} \dfrac{\partial g_1}{\partial z} & \dfrac{\partial g_1}{\partial w} \\ \dfrac{\partial h_1}{\partial z} & \dfrac{\partial h_1}{\partial w} \end{vmatrix} \tag{6-114}$$

Substituting the absolute value of (6-114) into (6-104), we get

$$f_{zw}(z, w) = \sum_i |J(z, w)| f_{xy}(x_i, y_i) = \sum_i \frac{1}{|J(x_i, y_i)|} f_{xy}(x_i, y_i) \tag{6-115}$$

since

$$|J(z, w)| = \frac{1}{|J(x_i, y_i)|} \tag{6-116}$$

where the determinant $J(x_i, y_i)$ represents the Jacobian of the original transformation in (6-99) given by

$$J(x_i, y_i) = \begin{vmatrix} \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \\ \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} \end{vmatrix}_{x=x_i, y=y_i} \tag{6-117}$$

We shall illustrate the usefulness of the formulas in (6-115) through various examples.

Linear Transformation

$$\mathbf{z} = a\mathbf{x} + b\mathbf{y} \qquad \mathbf{w} = c\mathbf{x} + d\mathbf{y} \tag{6-118}$$

If $ad - bc \neq 0$, then the system $ax + by = z$, $cx + dy = w$ has one and only one solution

$$x = Az + Bw \qquad y = Cz + Dw$$

Since $J(x, y) = ad - bc$, (6-115) yields

$$f_{zw}(z, w) = \frac{1}{|ad - bc|} f_{xy}(Az + Bw, Cz + Dw) \tag{6-119}$$

JOINT NORMALITY. From (6-119) it follows that if the random variables $\mathbf{x}$ and $\mathbf{y}$ are jointly normal as $N(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$ and

$$\mathbf{z} = a\mathbf{x} + b\mathbf{y} \qquad \mathbf{w} = c\mathbf{x} + d\mathbf{y} \tag{6-120}$$

then $\mathbf{z}$ and $\mathbf{w}$ are also jointly normal since $f_{zw}(z, w)$ will be an exponential (similar to $f_{xy}(x, y)$) with a quadratic exponent in z and w. Using the notation in (6-25), $\mathbf{z}$ and $\mathbf{w}$ in (6-120) are jointly normal as $N(\mu_z, \mu_w, \sigma_z^2, \sigma_w^2, \rho_{zw})$, where by direct computation

$$\mu_z = a\mu_x + b\mu_y$$
$$\mu_w = c\mu_x + d\mu_y$$
$$\sigma_z^2 = a^2\sigma_x^2 + 2ab\rho\sigma_x\sigma_y + b^2\sigma_y^2 \tag{6-121}$$
$$\sigma_w^2 = c^2\sigma_x^2 + 2cd\rho\sigma_x\sigma_y + d^2\sigma_y^2$$

and

$$\rho_{zw} = \frac{ac\sigma_x^2 + (ad + bc)\rho\sigma_x\sigma_y + bd\sigma_y^2}{\sigma_z\sigma_w}$$

In particular, *any* linear combination of two jointly normal random variables is normal.

EXAMPLE 6-22 ▶ Suppose $\mathbf{x}$ and $\mathbf{y}$ are zero mean independent Gaussian random variables with common variance σ^2. Define $\mathbf{r} = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$, $\boldsymbol{\theta} = \tan^{-1}(\mathbf{y}/\mathbf{x})$, where $|\theta| < \pi$. Obtain their joint density function.

SOLUTION
Here

$$f_{xy}(x, y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2} \tag{6-122}$$

Since

$$r = g(x, y) = \sqrt{x^2 + y^2} \qquad \theta = h(x, y) = \tan^{-1}(y/x) \tag{6-123}$$

and θ is known to vary in the interval $(-\pi, \pi)$, we have one solution pair given by

$$x_1 = r\cos\theta \qquad y_1 = r\sin\theta \tag{6-124}$$

We can use (6-124) to obtain $J(r, \theta)$. From (6-114)

$$J(r, \theta) = \begin{vmatrix} \dfrac{\partial x_1}{\partial r} & \dfrac{\partial x_1}{\partial \theta} \\ \dfrac{\partial y_1}{\partial r} & \dfrac{\partial y_1}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \tag{6-125}$$

so that

$$|J(r, \theta)| = r \tag{6-126}$$

We can also compute $J(x, y)$ using (6-117). From (6-123),

$$J(x, y) = \begin{vmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r} \tag{6-127}$$

Notice that $|J(r, \theta)| = 1/|J(x, y)|$, agreeing with (6-116). Substituting (6-122), (6-124) and (6-126) or (6-127) into (6-115), we get

$$f_{r,\theta}(r, \theta) = rf_{xy}(x_1, y_1) = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \qquad 0 < r < \infty \qquad |\theta| < \pi \tag{6-128}$$

Thus

$$f_r(r) = \int_{-\pi}^{\pi} f_{r,\theta}(r, \theta)\, d\theta = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \qquad 0 < r < \infty \tag{6-129}$$

which represents a Rayleigh random variable with parameter σ^2, and

$$f_\theta(\theta) = \int_0^\infty f_{r,\theta}(r, \theta)\, dr = \frac{1}{2\pi} \qquad |\theta| < \pi \tag{6-130}$$

which represents a uniform random variable in the interval $(-\pi, \pi)$. Moreover by direct computation

$$f_{r,\theta}(r, \theta) = f_r(r) \cdot f_\theta(\theta) \tag{6-131}$$

implying that $\mathbf{r}$ and $\boldsymbol{\theta}$ are independent. We summarize these results in the following statement: If $\mathbf{x}$ and $\mathbf{y}$ are zero mean independent Gaussian random variables with common variance, then $\sqrt{\mathbf{x}^2 + \mathbf{y}^2}$ has a Rayleigh distribution, and $\tan^{-1}(\mathbf{y}/\mathbf{x})$ has a uniform distribution in $(-\pi, \pi)$ (see also Example 6-15). Moreover these two derived random variables are statistically independent. Alternatively, with $\mathbf{x}$ and $\mathbf{y}$ as independent zero mean random variables as in (6-122), $\mathbf{x} + j\mathbf{y}$ represents a complex Gaussian random variable. But

$$\mathbf{x} + j\mathbf{y} = \mathbf{r}e^{j\theta} \tag{6-132}$$

with $\mathbf{r}$ and $\boldsymbol{\theta}$ as in (6-123), and hence we conclude that the magnitude and phase of a complex Gaussian random variable are independent with Rayleigh and uniform distributions respectively. The statistical independence of these derived random variables is an interesting observation. ◀

EXAMPLE 6-23 ▶ Let $\mathbf{x}$ and $\mathbf{y}$ be independent exponential random variables with common parameter λ. Define $\mathbf{u} = \mathbf{x} + \mathbf{y}$, $\mathbf{v} = \mathbf{x} - \mathbf{y}$. Find the joint and marginal p.d.f. of $\mathbf{u}$ and $\mathbf{v}$.

SOLUTION
It is given that

$$f_{xy}(x, y) = \frac{1}{\lambda^2} e^{-(x+y)/\lambda} \qquad x > 0 \qquad y > 0 \tag{6-133}$$

Now since $u = x + y$, $v = x - y$, always $|v| < u$, and there is only one solution given by

$$x = \frac{u + v}{2} \qquad y = \frac{u - v}{2} \tag{6-134}$$

Moreover the Jacobian of the transformation is given by

$$J(x, y) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

and hence

$$f_{uv}(u, v) = \frac{1}{2\lambda^2}e^{-u/\lambda} \qquad 0 < |v| < u < \infty \qquad (6\text{-}135)$$

represents the joint p.d.f. of **u** and **v**. This gives

$$f_u(u) = \int_{-u}^{u} f_{uv}(u, v)\, dv = \frac{1}{2\lambda^2}\int_{-u}^{u} e^{-u/\lambda}\, dv = \frac{u}{\lambda^2}e^{-u/\lambda} \qquad 0 < u < \infty \qquad (6\text{-}136)$$

and

$$f_v(v) = \int_{|v|}^{\infty} f_{uv}(u, v)\, du = \frac{1}{2\lambda^2}\int_{|v|}^{\infty} e^{-u/\lambda}\, du = \frac{1}{2\lambda}e^{-|v|/\lambda} \qquad -\infty < v < \infty$$

$$(6\text{-}137)$$

Notice that in this case $f_{uv}(u, v) \neq f_u(u) \cdot f_v(v)$, and the random variables **u** and **v** are not independent. ◀

As we show below, the general transformation formula in (6-115) making use of two functions can be made useful even when only one function is specified.

AUXILIARY VARIABLES. Suppose

$$\mathbf{z} = g(\mathbf{x}, \mathbf{y}) \qquad (6\text{-}138)$$

where **x** and **y** are two random variables. To determine $f_z(z)$ by making use of the formulation in (6-115), we can define an auxiliary variable

$$\mathbf{w} = \mathbf{x} \quad \text{or} \quad \mathbf{w} = \mathbf{y} \qquad (6\text{-}139)$$

and the p.d.f. of **z** can be obtained from $f_{zw}(z, w)$ by proper integration.

EXAMPLE 6-24　▶ Suppose $\mathbf{z} = \mathbf{x} + \mathbf{y}$ and let $\mathbf{w} = \mathbf{y}$ so that the transformation is one-to-one and the solution is given by $y_1 = w, x_1 = z - w$. The Jacobian of the transformation is given by

$$J(x, y) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

and hence

$$f_{zw}(x, y) = f_{xy}(x_1, y_1) = f_{xy}(z - w, w)$$

or

$$f_z(z) = \int f_{zw}(z, w)\, dw = \int_{-\infty}^{+\infty} f_{xy}(z - w, w)\, dw \qquad (6\text{-}140)$$

which agrees with (6-41). Note that (6-140) reduces to the convolution of $f_x(z)$ and $f_y(z)$ if **x** and **y** are independent random variables. ◀

Next, we consider a less trivial example along these lines.

EXAMPLE 6-25　▶ Let $\mathbf{x} \sim U(0, 1)$ and $\mathbf{y} \sim U(0, 1)$ be independent random variables. Define

$$\mathbf{z} = (-2\ln \mathbf{x})^{1/2}\cos(2\pi \mathbf{y}) \qquad (6\text{-}141)$$

Find the density function of **z**.

SOLUTION
We can make use of the auxiliary variable $\mathbf{w} = \mathbf{y}$ in this case. This gives the only solution to be

$$x_1 = e^{-[z\sec(2\pi w)]^2/2} \qquad (6\text{-}142)$$

$$y_1 = w \qquad (6\text{-}143)$$

and using (6-114)

$$J(z, w) = \begin{vmatrix} \dfrac{\partial x_1}{\partial z} & \dfrac{\partial x_1}{\partial w} \\[2ex] \dfrac{\partial y_1}{\partial z} & \dfrac{\partial y_1}{\partial w} \end{vmatrix} = \begin{vmatrix} -z\sec^2(2\pi w)e^{-[z\sec(2\pi w)]^2/2} & \dfrac{\partial x_1}{\partial w} \\[2ex] 0 & 1 \end{vmatrix}$$

$$= -z\sec^2(2\pi w)e^{-[z\sec(2\pi w)]^2/2} \qquad (6\text{-}144)$$

Substituting (6-142) and (6-144) into (6-115), we obtain

$$f_{zw}(z, w) = z\sec^2(2\pi w)e^{-[z\sec(2\pi w)]^2/2} \qquad -\infty < z < +\infty \quad 0 < w < 1 \qquad (6\text{-}145)$$

and

$$f_z(z) = \int_0^1 f_{zw}(z, w)\, dw = e^{-z^2/2}\int_0^1 z\sec^2(2\pi w)e^{-[z\tan(2\pi w)]^2/2}\, dw \qquad (6\text{-}146)$$

Let $u = z\tan(2\pi w)$ so that $du = 2\pi z\sec^2(2\pi w)\, dw$. Notice that as w varies from 0 to 1, u varies from $-\infty$ to $+\infty$. Using this in (6-146), we get

$$f_z(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}\underbrace{\int_{-\infty}^{\infty} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}}_{1} = \frac{1}{\sqrt{2\pi}}e^{-z^2/2} \qquad -\infty < z < \infty \qquad (6\text{-}147)$$

which represents a zero mean Gaussian random variable with unit variance. Thus $\mathbf{z} \sim N(0, 1)$. Equation (6-141) can be used as a practical procedure to generate Gaussian random variables from two independent uniformly distributed random sequences. ◀

EXAMPLE 6-26　▶ Let $\mathbf{z} = \mathbf{xy}$. Then with $\mathbf{w} = \mathbf{x}$ the system $xy = z, x = w$ has a single solution: $x_1 = w, y_1 = z/w$. In this case, $J(x, y) = -w$ and (6-115) yields

$$f_{zw}(z, w) = \frac{1}{|w|}f_{xy}\left(w, \frac{z}{w}\right)$$

Hence the density of the random variable $\mathbf{z} = \mathbf{xy}$ is given by

$$f_z(z) = \int_{-\infty}^{\infty} \frac{1}{|w|}f_{xy}\left(w, \frac{z}{w}\right)dw \qquad (6\text{-}148)$$

Special case: We now assume that the random variables **x** and **y** are independent and each is uniform in the interval $(0, 1)$. In this case, $z < w$ and

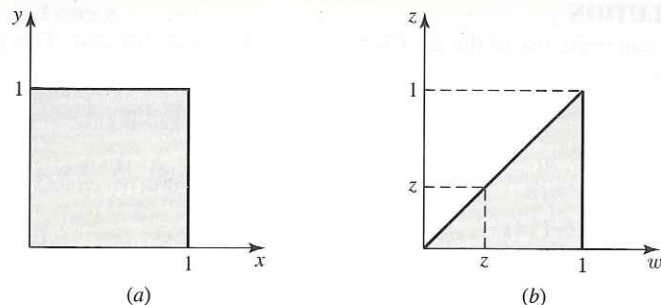$$f_{xy}\left(w, \frac{z}{w}\right) = f_x(w)f_y\left(\frac{z}{w}\right) = 1$$

(a)   (b)

FIGURE 6-25

so that (see Fig. 6-25)

$$f_{zw}(z, w) = \begin{cases} 1/w & 0 < z < w < 1 \\ 0 & \text{otherwise} \end{cases} \qquad (6\text{-}149)$$

Thus

$$f_z(z) = \int_z^1 \frac{1}{w} \, dw = \begin{cases} -\ln z & 0 < z < 1 \\ 0 & \text{elsewhere} \end{cases} \qquad (6\text{-}150)$$

◀

EXAMPLE 6-27 ▶ Let x and y be independent gamma random variables as in Example 6-12. Define $z = x + y$ and $w = x/y$. Show that z and w are independent random variables.

SOLUTION

Equations $z = x + y$ and $w = x/y$ generate one pair of solutions

$$x_1 = \frac{zw}{1+w} \qquad y_1 = \frac{z}{1+w}$$

Moreover

$$J(x, y) = \begin{vmatrix} 1 & 1 \\ 1/y & -x/y^2 \end{vmatrix} = -\frac{x+y}{y^2} = -\frac{(1+w)^2}{z}$$

Substituting these into (6-65) and (6-115) we get

$$f_{zw}(z, w) = \frac{1}{\alpha^{m+n} \Gamma(m)\Gamma(n)} \frac{z}{(1+w)^2} \left(\frac{zw}{1+w}\right)^{m-1} \left(\frac{z}{1+w}\right)^{n-1} e^{-z/\alpha}$$

$$= \frac{1}{\alpha^{m+n} \Gamma(m)\Gamma(n)} z^{m+n-1} e^{-z/\alpha} \cdot \frac{w^{m-1}}{(1+w)^{m+n}}$$

$$= \left(\frac{z^{m+n-1}}{\alpha^{m+n} \Gamma(m+n)} e^{-z/\alpha}\right) \cdot \left(\frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \frac{w^{m-1}}{(1+w)^{m+n}}\right)$$

$$= f_z(z) f_w(w) \qquad z > 0 \quad w > 0 \qquad (6\text{-}151)$$

showing that z and w are independent random variables. Notice that $z \sim G(m+n, \alpha)$ and w represents the ratio of two independent gamma random variables. ◀

EXAMPLE 6-28

THE STUDENT t DISTRIBUTION

▶ A random variable z has a Student t distribution[2] $t(n)$ with n degrees of freedom if for $-\infty < z < \infty$

$$f_z(z) = \frac{\gamma_1}{\sqrt{(1+z^2/n)^{n+1}}} \qquad \gamma_1 = \frac{\Gamma((n+1)/2)}{\sqrt{\pi n}\,\Gamma(n/2)} \qquad (6\text{-}152)$$

We shall show that if x and y are two independent random variables, x is $N(0, 1)$, and y is $\chi^2(n)$:

$$f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad f_y(y) = \frac{1}{2^{n/2}\Gamma(n/2)} y^{n/2-1} e^{-y/2} U(y) \qquad (6\text{-}153)$$

then the random variable

$$z = \frac{x}{\sqrt{y/n}}$$

has a $t(n)$ distribution. Note that the Student t distribution represents the ratio of a normal random variable to the square root of an independent χ^2 random variable divided by its degrees of freedom.

SOLUTION

We introduce the random variable $w = y$ and use (6-115) with

$$x = z\sqrt{\frac{w}{n}} \qquad y = w \qquad J(z, w) = \sqrt{\frac{w}{n}} \quad \text{or} \quad J(x, y) = \sqrt{\frac{n}{w}}$$

This yields

$$f_{zw}(z, w) = \sqrt{\frac{w}{n}} \frac{1}{\sqrt{2\pi}} e^{-z^2 w/2n} \frac{w^{n/2-1}}{2^{n/2}\Gamma(n/2)} e^{-w/2} U(w)$$

$$= \frac{w^{(n-1)/2}}{\sqrt{2\pi n}\, 2^{n/2}\Gamma(n/2)} e^{-w(1+z^2/n)/2} U(w)$$

Integrating with respect to w after replacing $w(1 + z^2/n)/2 = u$, we obtain

$$f_z(z) = \frac{1}{\sqrt{\pi n}\,\Gamma(n/2)} \frac{1}{(1+z^2/n)^{(n+1)/2}} \int_0^\infty u^{(n-1)/2} e^{-u} \, du$$

$$= \frac{\Gamma((n+1)/2)}{\sqrt{\pi n}\,\Gamma(n/2)} \frac{1}{(1+z^2/n)^{(n+1)/2}}$$

$$= \frac{1}{\sqrt{n}\,\beta(1/2, n/2)} \frac{1}{(1+z^2/n)^{(n+1)/2}} \qquad -\infty < z < \infty \qquad (6\text{-}154)$$

For $n = 1$, (6-154) represents a Cauchy random variable. Notice that for each n, (6-154) generates a different p.d.f. As n gets larger, the t distribution tends towards the normal distribution. In fact from (6-154)

$$(1 + z^2/n)^{-(n+1)/2} \to e^{-z^2/2} \quad \text{as} \quad n \to \infty$$

For small n, the t distributions have "fatter tails" compared to the normal distribution because of its polynomial form. Like the normal distribution, Student t distribution is important in statistics and is often available in tabular form. ◀

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[2] *Student* was the pseudonym of the English statistician W. S. Gosset, who first introduced this law in empirical form (*The probable error of a mean,* Biometrica, 1908.) The first rigorous proof of this result was published by R. A. Fisher.

EXAMPLE 6-29

THE F DISTRI-BUTION

▶ Let $\mathbf{x}$ and $\mathbf{y}$ be independent random variables such that $\mathbf{x}$ has a chi-square distribution with m degrees of freedom and $\mathbf{y}$ has a chi-square distribution with n degrees of freedom. Then the random variable

$$\mathbf{F} = \frac{\mathbf{x}/m}{\mathbf{y}/n} \qquad (6\text{-}155)$$

is said to have an F distribution with (m, n) degrees of freedom. Show that the p.d.f. of $\mathbf{z} = \mathbf{F}$ is given by

$$f_z(z) = \begin{cases} \dfrac{\Gamma((m+n)/2)m^{m/2}n^{n/2}}{\Gamma(m/2)\Gamma(n/2)} \dfrac{z^{m/2-1}}{(n+mz)^{(m+n)/2}} & z > 0 \\ 0 & \text{otherwise} \end{cases} \qquad (6\text{-}156)$$

SOLUTION

To compute the density of $\mathbf{F}$, using (6-153) we note that the density of $\mathbf{x}/m$ is given by

$$f_1(x) = \begin{cases} \dfrac{m(mx)^{m/2-1}e^{-mx/2}}{\Gamma(m/2)2^{m/2}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and that of $\mathbf{y}/n$ by

$$f_2(y) = \begin{cases} \dfrac{n(ny)^{n/2-1}e^{-ny/2}}{\Gamma(n/2)2^{n/2}} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Using (6-60) from Example 6-10, the density of $\mathbf{z} = \mathbf{F}$ in (6-155) is given by

$$f_z(z) = \int_0^\infty y \left(\frac{m(mzy)^{m/2-1}e^{-mzy/2}}{\Gamma(m/2)2^{m/2}} \right) \left(\frac{n(ny)^{n/2-1}e^{-ny/2}}{\Gamma(n/2)2^{n/2}} \right) dy$$

$$= \frac{(m/2)^{m/2}(n/2)^{n/2}}{\Gamma(m/2)\Gamma(n/2)2^{(m+n)/2}} z^{m/2-1} \int_0^\infty y^{(m+n)/2-1} e^{y(n+mz)/2} \, dy$$

$$= \frac{(m/2)^{m/2}(n/2)^{n/2}}{\Gamma(m/2)\Gamma(n/2)2^{(m+n)/2}} z^{m/2-1} \Gamma\left(\frac{m+n}{2}\right) \left(\frac{2}{n+mz} \right)^{(m+n)/2}$$

$$= \frac{\Gamma((m+n)/2)m^{m/2}n^{n/2}}{\Gamma(m/2)\Gamma(n/2)} \frac{z^{m/2-1}}{(n+mz)^{(m+n)/2}}$$

$$= \frac{(m/n)^{m/2}}{\beta(m/2, n/2)} z^{m/2-1}(1+mz/n)^{-(m+n)/2} \qquad z > 0 \qquad (6\text{-}157)$$

and $f_z(z) = 0$ for $z \le 0$. The distribution in (6-157) is called Fisher's variance ration distribution. If $m = 1$ in (6-155), then from (6-154) and (6-157) we get $\mathbf{F} = [t(n)]^2$. Thus $F(1, n)$ and $t^2(n)$ have the same distribution. Moreover $F(1, 1) = t^2(1)$ represents the square of a Cauchy random variable. Both Student's t distribution and Fisher's F distribution play key roles in statistical tests of significance. ◀

6-4 JOINT MOMENTS

Given two random variables $\mathbf{x}$ and $\mathbf{y}$ and a function $g(x, y)$, we form the random variable $\mathbf{z} = g(\mathbf{x}, \mathbf{y})$. The expected value of this random variable is given by

$$E\{\mathbf{z}\} = \int_{-\infty}^{\infty} z f_z(z) \, dz \qquad (6\text{-}158)$$

However, as the next theorem shows, $E\{\mathbf{z}\}$ can be expressed directly in terms of the function $g(x, y)$ and the joint density $f(x, y)$ of $\mathbf{x}$ and $\mathbf{y}$.

THEOREM 6-4 ▶

$$E\{g(\mathbf{x}, \mathbf{y})\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) \, dx \, dy \qquad (6\text{-}159)$$

Proof. The proof is similar to the proof of (5-55). We denote by ΔD_z the region of the xy plane such that $z < g(x, y) < z + dz$. Thus to each differential in (6-158) there corresponds a region ΔD_z in the xy plane. As dz covers the z axis, the regions ΔD_z are not overlapping and they cover the entire xy plane. Hence the integrals in (6-158) and (6-159) are equal. ◀

We note that the expected value of $g(\mathbf{x})$ can be determined either from (6-159) or from (5-55) as a single integral

$$E\{g(\mathbf{x})\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) f(x, y) \, dx \, dy = \int_{-\infty}^{\infty} g(x) f_x(x) \, dx$$

This is consistent with the relationship (6-10) between marginal and joint densities.

If the random variables $\mathbf{x}$ and $\mathbf{y}$ are of discrete type taking the values x_i and y_k with probability p_{ik} as in (6-33), then

$$E\{g(\mathbf{x}, \mathbf{y})\} = \sum_i \sum_k g(x_i, y_k) p_{ik} \qquad (6\text{-}160)$$

Linearity From (6-159) it follows that

$$E\left\{ \sum_{k=1}^n a_k g_k(\mathbf{x}, \mathbf{y}) \right\} = \sum_{k=1}^n a_k E\{g_k(\mathbf{x}, \mathbf{y})\} \qquad (6\text{-}161)$$

This fundamental result will be used extensively.

We note in particular that

$$E\{\mathbf{x} + \mathbf{y}\} = E\{\mathbf{x}\} + E\{\mathbf{y}\} \qquad (6\text{-}162)$$

Thus the expected value of the sum of two random variables equals the sum of their expected values. We should stress, however, that, in general,

$$E\{\mathbf{xy}\} \ne E\{\mathbf{x}\}E\{\mathbf{y}\}$$

Frequency interpretation As in (5-51)

$$E\{\mathbf{x} + \mathbf{y}\} \simeq \frac{\mathbf{x}(\xi_1) + \mathbf{y}(\xi_i) + \cdots + \mathbf{x}(\xi_n) + \mathbf{y}(\xi_n)}{n}$$

$$= \frac{\mathbf{x}(\xi_1) + \cdots + \mathbf{x}(\xi_n)}{n} + \frac{\mathbf{y}(\xi_1) + \cdots + \mathbf{y}(\xi_n)}{n}$$

$$\simeq E\{\mathbf{x}\} + E\{\mathbf{y}\}$$