Capacity-approaching LDPC Codes Based on Markov Chain Monte Carlo MIMO Detection

Rong-Rong Chen and Behrouz Farhang-Boroujeny
Department of Electrical and Computer Engineering,
University of Utah, UT 84112
{rchen,farhang}@ece.utah.edu

Alexei Ashikhmin
Bell Laboratories, Lucent Technologies,
Murray Hill, NJ 07974
aea@bell-labs.com

***Abstract*— In this work we study joint code design and MIMO (multiple-input multiple-output) detection based on Markov Chain Monte Carlo (MCMC) approach. The extrinsic information transfer (EXIT) technique is applied to find optimal irregular LDPC codes that are matched to the characteristics of the MCMC detector. For a large system of 8 transmit and 8 receive antennas and 16 QAM modulation, the optimized LDPC code achieves within 1.4 dB of the channel capacity. This result improves best published results by 2.3 dB, where 1.3 dB is due to the MCMC detection and 1 dB is due to code optimization.**

I. Introduction

In recent years, the use of multiple antennas at both transmitter and receiver ends of a communication link has been identified and widely studied as the most practical method of increasing the channel capacity, and also to guarantee a reliable link (i.e., to minimize the outage probability) [1], [2], [3], [4]. Such systems can theoretically increase capacity by up to a factor equaling the minimum of the number of transmit and receive antennas in the array [1], [2]. The term MIMO (multiple-input multiple-output) is used to refer to such channels.

While MIMO channels promise significant capacity gain, it is a challenging task to design capacity-approaching codes and detection strategies to realize the promised capacity gain. It is well-known that powerful channel codes such as turbo codes [5], [6] and low density parity check (LDPC) codes [7], [8] can approach channel capacity for standard additive white Gaussian noise (AWGN) channels. Hence, these codes offer hope to bring us close to capacity for nonstandard channels including multiple antenna fading channels. For such channels, however, powerful channel codes alone are not sufficient. At the receiver end, a MIMO detector should be used to separate the data symbols that have been transmitted concurrently. Most importantly, in order to achieve capacity, the receiver should resort to "turbo-principle" to allow iterative exchange of soft information between the MIMO detector and the channel decoder. Such iterative receivers are necessary to facilitate near capacity performance at low-complexity.

It is shown in [16], [15], [14] that MIMO detectors based on the Gibbs sampler/ Markov chain Monte Carlo (MCMC) methods lead to significant performance gain over other detection methods including those based on sphere decoding. In this work, we investigate the application of MCMC detection in the design of capacity-approaching LDPC codes. We optimize the irregular LDPC code to match the characteristics of the MCMC detector based on the extrinsic information transfer (EXIT) methods [10], [12]. Our results show that the MCMC approach leads to feasible design of channel codes that indeed perform very close to the capacity limit. Compared to the best published results [13], we obtain a performance improvement of 2.3 dB, where 1.3 dB is due to the MCMC detection and 1 dB is due to LDPC code optimization. This demonstrates the effectiveness of joint code design and MCMC detection for MIMO channels.

II. System Model

We consider a MIMO channel with t transmit and r receive antennas. The channel model is given by :

$$\mathbf{Y} = \sqrt{\frac{\rho}{t}}\mathbf{H}\mathbf{D} + \mathbf{N}, \tag{1}$$

where $\mathbf{D} \in \mathbb{C}^t$, $\mathbf{Y} \in \mathbb{C}^r$, and $\mathbf{N} \in \mathbb{C}^r$ are complex column vectors that represent the transmitted signal, received signal, and channel noise respectively; $\mathbf{H}$ is the r by t channel fading matrix with i.i.d. Rayleigh fading entries; the noise vector $\mathbf{N}$ has i.i.d. complex Gaussian entries with variance of unity; ρ is the signal-to-noise ratio per receive antenna. Throughout this paper, we use capital letters to represent random variables and lower case letters to represent realizations of the random variables. We also use bold letters to represent vectors and matrices.

Figure 1 shows a schematic block diagram of the MIMO system. The channel encoder encodes a sequence of information bits $\mathbf{b}$ into a sequence of coded bits. The resulting coded sequence is interleaved by a random permutation indicated by π and then mapped to a sequence of QAM symbols $\mathbf{d}$ using Gray mapping. The sequence $\mathbf{d}$ is then divided into blocks of t symbols and sent through t transmit antennas over the MIMO fading channel. At the receiver end, for each set of r received signal samples from the r receive antennas, the MIMO detector computes *a posteriori* probabilities (APPs) of the input

QAM symbols. Subsequently, these symbol APPs are used to compute bit-wise APPs. The "extrinsic" part of these bit-wise APPs is then de-interleaved and passed to the channel decoder. The channel decoder performs one or more decoding iterations and generates bit-wise extrinsic information. Assuming that the bits which constitute a QAM symbol are statistically independent, the interleaved bit-wise extrinsic information $\{\lambda^b\}$ is fed back to the MIMO detector, which updates the prior symbol probabilities λ^s (details of converting symbol-wise APPs and bit-wise APPs are given in Section III). For the next iteration, the detector computes symbol APPs using λ^s. In this manner, the MIMO detection and channel decoding proceeds iteratively. After a fixed number of iterations, decisions are made at the output of the channel decoder to generate the decoded bits $\hat{\mathbf{b}}$.

III. MIMO DETECTOR BASED ON MARKOV CHAIN MONTE CARLO (MCMC) / GIBBS SAMPLER

In this section, we briefly review MCMC detector proposed in [14], [15], [16]. Due to the statistical independence of the fading matrices $\mathbf{H}$ over time, the MIMO detector operates independently for each set of r received signals. We assume that $\mathbf{H}$ is perfectly known at the receiver.

Let $\mathbf{y} = [y_1, \cdots, y_t]^T$ be a set of received signals. Here $(\cdot)^T$ represents the transpose operator. Let $\mathbf{D}_{-i} = [D_1 \ \ldots \ D_{i-1} \ D_{i+1} \ \cdots D_t]^T$ denote the set of transmitted QAM symbols except for the i-th symbol. Let λ^s denote the symbol-wise prior probabilities computed based on bit-wise APPs coming from the channel decoder:

$$\lambda^s(D_i = a) = \prod_{j=1}^{M_c} \lambda^b(D_{ij} = a_j), \qquad (2)$$

where M is the constellation size; $M_c = \log_2 M$ is the number of bits in each QAM symbol; D_{ij} is the j-th bit of symbol D_i; a is an arbitrary symbol from the QAM constellation and a_j is its j-th bit. Using λ^s, the MIMO detector computes updated symbol-wise APPs β^s as follows:

$$\begin{aligned}
&\beta^s(D_i = a) = P(D_i = a|\mathbf{y}; \lambda^s) \\
&= \sum_{\mathbf{d}_{-i}} P\left(D_i = a, \mathbf{D}_{-i} = \mathbf{d}_{-i}|\mathbf{y}, \lambda^s\right) \\
&\propto \sum_{\mathbf{d}_{-i}} p(\mathbf{y}|D_i = a, \mathbf{D}_{-i} = \mathbf{d}_{-i}) \cdot P(D_i = a, \mathbf{D}_{-i} = \mathbf{d}_{-i}|\lambda^s) \\
&= \sum_{\mathbf{d}_{-i}} p(\mathbf{y}|D_i = a, \mathbf{D}_{-i} = \mathbf{d}_{-i}) \cdot \lambda^s(D_i = a) \\
&\qquad \cdot \lambda^s(\mathbf{D}_{-i} = \mathbf{d}_{-i}), \quad i = 1, \cdots, t. \qquad (3)
\end{aligned}$$

Note that the summation in (3) is over *all* possible choices of $\mathbf{d}_{-i}$, therefore the total number of terms in the summation equals M^{t-1}. This follows that the complexity of the optimal detector (corresponding to exact computation of the summation) is exponential with respect to M.

In order to reduce complexity, we approximate the optimal detector (3) by

$$\begin{aligned}
\beta^s(D_i = a) \quad \approx \quad & k \cdot \sum_{\mathbf{d}_{-i} \in \mathcal{C}_i} p(\mathbf{y}|D_i = a, \mathbf{D}_{-i} = \mathbf{d}_{-i}) \\
& \cdot \lambda^s(D_i = a) \cdot \lambda^s(\mathbf{D}_{-i} = \mathbf{d}_{-i}), \qquad (4)
\end{aligned}$$

where the summation is over some sample set $\mathcal{C}_i$, and k is a normalization constant to ensure that $\sum_a \beta^s(D_i = a) = 1$. Clearly, in order for (4) to be a good approximation of the optimal detector, the samples in $\mathcal{C}_i$ should contain a sufficient number of the "significant" terms in the summation of (3). Once the symbol-wise APPs β^s are computed, the bit-wise APPs can be computed using

$$P(D_{ij} = b) = \sum_{a: a_j = b} \beta^s(D_i = a), \qquad b = 0, 1.$$

Subsequently, the "extrinsic part" of these bit-wise APPs are passed back to the channel decoder.

In the following, we state the statistical procedure for finding the sample set $\mathcal{C}_i$ based on the MCMC method [14], [15], [16].

Gibbs sampler:
$n = 0$; Generate the initial $\mathbf{d}^{(0)}$ vector using the priors λ^s.
% In the following for-loop sample vector $\mathbf{d}^{(n)}$ is generated
% by changing/sampling one symbol at a time.
for $n = 1$ to T
  generate $d_1^{(n)}$ using the distribution
$P(D_1 = a|d_2^{(n-1)}, d_3^{(n-1)}, \cdots, d_t^{(n-1)}, \mathbf{y}, \boldsymbol{\lambda}^s)$
  generate $d_2^{(n)}$ using the distribution
$P(D_2 = a|d_1^{(n)}, d_3^{(n-1)}, \cdots, d_t^{(n-1)}, \mathbf{y}, \boldsymbol{\lambda}^s)$
  ⋮
  generate $d_t^{(n)}$ using the distribution
$P(D_t = a|d_1^{(n)}, d_2^{(n)}, \cdots, d_{t-1}^{(n)}, \mathbf{y}, \boldsymbol{\lambda}^s)$
%end for-loop

Note that in the step of generating sample $d_i^{(n)}$, only the prior probability for symbol D_i is needed. For instance,

$$\begin{aligned}
&P(D_1 = a|d_2^{(n-1)}, d_3^{(n-1)}, \cdots, d_t^{(n-1)}, \mathbf{y}, \lambda^s) \\
&\propto p(\mathbf{y}|D_1 = a, d_2^{(n-1)}, \cdots, d_t^{(n-1)}) \cdot \lambda^s(D_1 = a),
\end{aligned}$$

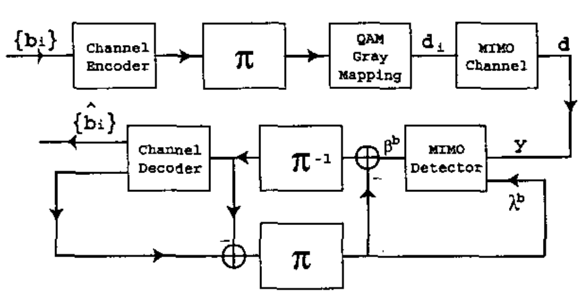

Fig. 1. A schematic block diagram of the MIMO system.

which involves only $\lambda^s(D_1 = a)$.

Following the procedure above, the Gibbs sampler runs over all symbols T times to generate a collection of vectors $\mathbf{d}^{(1)}, \cdots, \mathbf{d}^{(T)}$ which (together with the initial vector $\mathbf{d}^{(0)}$) are put into a set $\mathcal{C}$. In [16], it is noted that obtaining Gibbs samples from a number of independent/parallel Gibbs samplers results in more robust detectors. Here, also, we run Q parallel Gibbs sampler to get more samples and they are all added (except for repetitions) to the set $\mathcal{C}$. The set $\mathcal{C}_i$ used in equation (4) consists of all samples in $\mathcal{C}$ with their i-th symbol deleted. Since we require that all vectors in $\mathcal{C}_i$ are distinct, the size of $\mathcal{C}_i$ is no more than $Q \cdot (T+1)$.

IV. Optimized irregular LDPC codes for the MCMC detector

In order to find optimal LDPC codes that are matched to the MCMC detector, we follow code design methods proposed by S. ten Brink *et. al.* [12]. In [12], the optimal MIMO detector is employed. Hence, due to the exponential complexity of the optimal detector, capacity-approaching irregular LDPC codes could only be found for MIMO systems with small number of antennas (up to 4 antennas) and small constellation sizes (QPSK). Here, we show that replacement of the optimal MIMO detector by a MCMC detector allows feasible extension of the results of [12] to larger systems, e.g., a MIMO system with 8 transmit and 8 receive antennas and 16 QAM modulation.

In [12], it is shown that optimal code parameters can be found by matching two EXIT curves: the combined detector and variable node (DET/VND) curve and the check node (CND) curve. In order to compute the DET/VND curve, we first obtain EXIT curves of the MCMC detector given a finite number of Gibbs samples. Moreover, we assume that each Markov chain runs for T iterations of Gibbs sampler and we run Q Gibbs samplers in parallel. Larger values of Q and T, clearly, result in better detector performance (corresponding to higher EXIT curves). Figure 2 shows a set of such curves. Once the detector EXIT curve is determined, following [12], linear programming techniques are applied to find optimal code parameters that match the DET/VND and CND curves. As shown in Figure 3, using the detector curve for $Q = 20, T = 10$, the optimized code matches the two EXIT curves very well at $E_b/N_0 = 4$ dB, which is slight above the capacity limit of 3.8 dB.

In Figure 4, we consider a system with 8 transmit and 8 receive antennas and 16 QAM modulation.

Note that we use the normalized SNR E_b/N_0 as

$$\left.\frac{E_b}{N_0}\right|_{dB} = \left.\frac{E_s}{N_0}\right|_{dB} + 10\log 10\left(\frac{T}{RtM_c}\right), \tag{5}$$

where $\frac{E_s}{N_0} = \rho$ is the received signal-to-noise ratio per receive antenna and R is the rate of the channel code. Here we assume that the code rate is $R = 1/2$ and the code length is 18432 (corresponding to 9216 information bits).

Three sets of simulation results are presented. The right most curve is the performance curve in [13] that uses a turbo code and a sphere decoding based MIMO detector. Our results show that by replacing the MIMO detector in [13] with the MCMC detector and using the same turbo code, we obtain a performance gain of about 1 dB, shown as the second curve from the right. Note that the MCMC detector we use has small parameters, e.g., $Q = T = 5$. Hence, only 30 samples are used, while in [13] a total of 1024 samples found by the sphere decoding algorithm are used to compute symbol APPs. Furthermore, we show that if we replace the turbo code with an optimized LDPC code that is matched to the MCMC detector, we obtain an additional gain of 1.3 dB. The three leftmost curves are performance curves of the optimized LDPC code in conjunction with MCMC detectors of various parameters. As shown, the best performance is achieved by using the MCMC detector with larger parameters $Q = 20$, $T = 10$ and it achieves within 1.4 dB of the capacity limit (3.8 dB) at BER of 10^{-5}. It is important to note that the MCMC detector with smaller parameters, e.g., $Q = T = 5$ induces a performance degradation

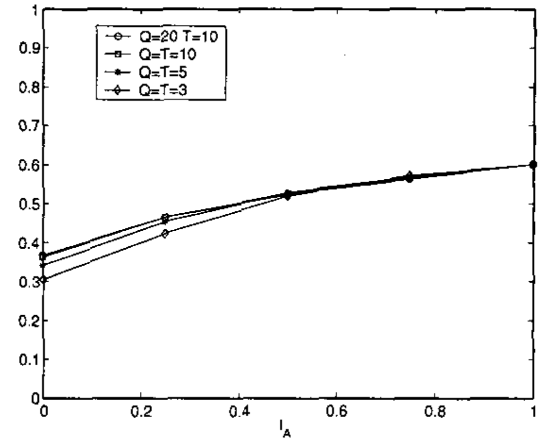

Fig. 2. EXIT curves of the MCMC detector at $E_b/N_0 = 4$ dB. This is for 8 transmit and 8 receive antennas and 16 QAM modulation. From top to bottom, the curves correspond to $Q = 20, T = 10$, $Q = T = 10$, $Q = T = 5$, and $Q = T = 3$, respectively.

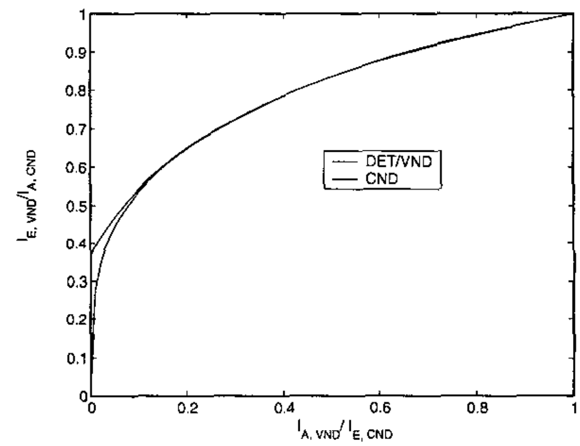

Fig. 3. Optimized LDPC code using the $Q = 20, T = 10$ detector EXIT curve. The optimized code matches the DET/VND and CND curves at $E_b/N_0 = 4$ dB. All check nodes have the same degree $d_c = 5$. Among all variable nodes, 68.9% have degree 2, 21.6% have degree 3, and 9.5% have degree 5.

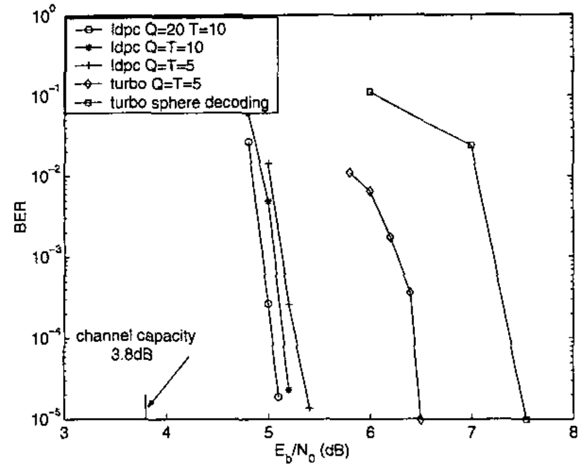

Fig. 4. BER results for a MIMO system with 8 transmit and 8 receive antennas and 16 QAM modulation. The rightmost curve is the performance curve of a turbo code with sphere decoding of sample size 1024 [13]. The second curve from the right is the performance curve of the same turbo code with a MCMC detector of parameters $Q = T = 5$. The left three curves (from left to right) are performance curves of the optimized LDPC code with MCMC detectors of parameters $Q = 20, T = 10$, $Q = T = 10$, and $Q = T = 5$, respectively.

of only 0.3 dB.

We have also simulated performance of the 4 transmit and 4 receive antenna system with 16 QAM modulation, using an optimized LDPC code that is matched to the MCMC detector (results are not shown here). For this system we also achieve within 1.4 dB of the capacity limit. This gives a performance gain of about 1.3 dB compared to the best results reported in [13].

In summary, several factors contribute to the significant performance improvement of this work over [13]. First, the performance of the MCMC detector is superior than that of the sphere decoding detector adopted in [13]. Detailed comparisons between these detectors can be found in [17]. Second, we have conducted LDPC code optimization such that the channel code is optimally matched to the MCMC detector. Without the LDPC code optimization, as shown in Figure 4, the performance gain reduces to about 1 dB. Third, while in [13] the suboptimal max-log-MAP algorithm is used, in our simulations we use the more accurate log-MAP algorithm to compute (4). This is made possible due to the fact that only a small number of samples is required by the MCMC detector, as opposed to a much larger number of samples required by the sphere decoder.

V. Conclusion

In this paper, we presented our results on joint LDPC code design and MCMC detection for large antenna systems. The optimized irregular LDPC code in conjunction with MCMC detection improves best published results significantly and achieves within 1.4 dB of the channel capacity. The number of samples required by the MCMC detector is shown to be much less than that of the MIMO detector based on sphere decoding. This confirms the effectiveness of the MCMC approach in MIMO detection and code design.

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