Ex:

\[ X_i(s) \rightarrow 4 \rightarrow + \rightarrow \frac{1}{s+2} \rightarrow X_0(s) \]

\[ \frac{K}{s+3} \rightarrow \frac{1}{s+1} \]

a) Find the transfer function, \( H(s) = \frac{X_0(s)}{X_i(s)} \), for the above system.

b) For what values of \( K \) is the system on the verge of being unstable? (Hint: the system is on the verge of being unstable when one of the characteristic roots is at \( s = 0 \).)

\[ \text{sol'n: a) The box out front multiplies the transfer function of the rest of the system by 4.} \]

\[ 4X_i \rightarrow + \rightarrow \frac{1}{s+2} \rightarrow X_0 \]

For the rest of the system we may combine the boxes in the feedback into a single box.

\[ \frac{K}{s+3} \cdot \frac{1}{s+1} \]

For this configuration, the transfer function is

\[ \frac{X_0(s)}{4X_i(s)} = \frac{A}{1 + AB} = \frac{1}{\frac{s+2}{s+3}(s+1)} \]

So

\[ H(s) = \frac{X_0(s)}{X_i(s)} = \frac{4}{\frac{s+2}{(s+1)(s+2)(s+3)}} \]
b) We find the characteristic roots by writing $H(s)$ as a ratio of polynomials in $s$ and setting the denominator to zero.

$$H(s) = \frac{4}{s+2} \cdot \frac{(s+1)(s+2)(s+3)}{1 + K \cdot \frac{(s+1)(s+2)(s+3)}{(s+1)(s+2)(s+3)}}$$

Note that we multiply on top and bottom by the denominator of the denominator.

$$H(s) = \frac{4(s+1)(s+3)}{(s+1)(s+2)(s+3) + K}$$

Setting the denominator equal to zero gives:

$$(s+1)(s+2)(s+3) + K = 0$$

or

$$(s+1)(s^2 + 5s + 6) + K = 0$$

or

$$s^3 + 6s^2 + 11s + 6 + K = 0$$

To have a root at $s=0$, we must have a polynomial from which can factor out $(s-0)$ or $s$. This means we need to eliminate the constant term, $6 + K$.

$$6 + K = 0 \quad \text{or} \quad K = -6$$

Note: We may verify that the other roots are stable when $K = -6$:

$$s^3 + 6s^2 + 11s = s(s + 3 + j\sqrt{2})(s + 3 - j\sqrt{2})$$

So roots are $s=0$, $s = -3-j\sqrt{2}$, and $s = -3+j\sqrt{2}$. The real parts are less than or equal to 0.