a) Calculate roots of characteristic eq'n for \( v(t) \).

b) Is response over-, under-, or critically damped?

c) What \( R \) yields damped freq at 6 kHz rad/s?

d) Find \( \lambda \) of characteristic eq'n for \( R \) found in (c).

e) Find \( R \) for critically damped response.

\[ \text{Ans: a)} \quad s_1 = -5k / s, \quad s_2 = -20k / s \quad \text{(units are actually kHz rad/s)} \\
\text{b)} \quad \text{overdamped} \\
\text{c)} \quad R = 7.8125 \ \text{k}\Omega \\
\text{d)} \quad s_1 = -8k + j6k / s, \quad s_2 = -8k - j6k / s \quad \text{(actually kHz rad/s)} \\
\text{e)} \quad R = 6.25 \ \text{k}\Omega \]

\[ \text{Sln: a)} \quad \text{From Kirchhoff's law, we know the total current } \\
\text{flowing out of the top center node is zero:} \\
\]

\[ i_R + i_L + i_C = 0A \]

\[ \text{We calculate each of the currents:} \]

\[ i_R = \frac{v}{R} \quad i_L = ? \quad \text{We have to start with } \frac{v}{L} \frac{di_L}{dt} \]

\[ \text{and solve for } i_L. \]

\[ \text{Multiply both sides of } v = L \frac{di_L}{dt} \text{ by } dt \text{ and integrate:} \]

\[ \int_{t' = 0}^{t} L \frac{di_L}{dt} \cdot dt' = \int_{t' = 0}^{t} \frac{v}{L} \cdot dt' \quad \text{we use } t' \text{ as the dummy variable of integration to avoid confusion with time } t. \]

\[ \text{Furthermore, the lower and upper limits of integration to the type of differential: } t' \text{ for } dt', \quad i_L(t) \text{ for } di_L. \]

\[ \text{The limits of integration to the } t' \text{ must be evaluated at the same points in time: } t' = 0 \text{ and } i_L(t = 0), \text{ and } t = t' \text{ and } i_L(t'). \]
The right-hand side of the preceding equation simplifies:

\[
\int_{t'=0}^{t'=t} r \, dt' = \int_L \frac{di_L}{i(t)=i_L(t=0)}
\]

Solving for \( i_L(t) \) by moving the \( i_L(t=0) \) to the other side gives:

\[
i_L(t) = \frac{1}{L} \int_{t'=0}^{t'=t} r \, dt' + i_L(t=0)
\]

Finally, for \( i_C \), we use \( i_C = C \frac{dv}{dt} \).

For \( i_R + i_L + i_C = 0 \) we have:

\[
\frac{1}{R} \int_{t'=0}^{t'=t} v \, dt' + i_L(t=0) + C \frac{dv}{dt} = 0 \, A
\]

To eliminate the integral, we differentiate both sides. (This also gets rid of \( i_L(t=0) \), which is a constant.)

\[
\frac{1}{R} \frac{dv}{dt} + \frac{1}{L} v + C \frac{d^2 v}{dt^2} = 0 \, A/s \quad \text{(at time } t)\]

Note that \( \frac{d}{dt} \int_{t'=0}^{t'=t} v \, dt' = v(t) \) because the rate of change of the integral of \( v \) at time \( t \) is just \( v(t) \):

\[\text{change in } \int v \, dt' \text{ in time } dt' \text{ at absolute time } t \]

The rate of change of \( \int v \, dt' \) at time \( t \) is

\[
\lim_{dt' \to 0} \frac{1}{dt'} v(t) \cdot dt' = \lim_{dt' \to 0} v(t) = v(t).
\]
Now try sol'n \( v(t) = Ae^{st} \) in diff. eq'n:

\[
\frac{dv}{dt} = AESe^{st} \\
\frac{d^2v}{dt^2} = A\frac{d^2}{dt^2}e^{st}
\]

\[
\frac{1}{R}Ase^{st} + \frac{1}{L}Ae^{st} + CAs^2e^{st} = 0 \quad \text{A/s}
\]

Note: \( A \) is some constant we must determine, and \( s \) is another """""". We determine \( s \) from the characteristic eq'n we obtain by factoring \( Ae^{st} \) out of the above eq'n:

\[
\left( \frac{1}{R}S + \frac{L}{C} + Cs^2 \right) Ae^{st} = 0 \quad \text{A/s}
\]

Clearly, either \( Ae^{st} = 0 \) or \( \frac{1}{R}S + \frac{L}{C} + Cs^2 = 0 \).

If \( Ae^{st} = 0 \) then \( v(t) = 0 \) always. This is impossible since we can build an RLC and observe that \( v(t) \neq 0 \). We conclude that, if \( Ae^{st} \) does indeed work as a sol'n,

\[
\frac{1}{R}S + \frac{L}{C} + Cs^2 = 0. \quad \text{This is the characteristic eq'n.}
\]

This is a quadratic eq'n that we put in a convenient form by dividing thru by \( C \):

\[
\frac{s^2}{RC} + \frac{L}{LC} = 0
\]

We also define a convenient notation:

\[
s^2 + 2\alpha s + \omega_0^2 = 0 \quad \kappa = \frac{1}{2RC} \quad \omega_0 = \frac{1}{\sqrt{LC}}
\]

We solve for \( s \) and discover there are two sol'ns:

\[
s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}
\]
case I: If \( \alpha > \omega_0 \) we get \( s_1, s_2 \) real and our \( v(t) \) will consist of the sum of two decaying exponentials. (Note that \( s_1, s_2 < 0 \) so \( e^{s_1t} \) and \( e^{s_2t} \) decay.)

\[
v(t) = A_1e^{s_1t} + A_2e^{s_2t} \quad \text{overdamped}
\]

case II: If \( \alpha = \omega_0 \) we get \( s_1 = s_2 = \alpha \) real and our \( v(t) \) will consist of the sum of a decaying exponential and \( t \) times that same decaying exponential. (Note that we have not explained why this happens, but we get this result if we take the limit of the case I solution as \( s_1 - s_2 \to 0 \).)

\[
v(t) = (D_1 + D_2) e^{-\alpha t} \quad \text{critically damped}
\]

case III: If \( \alpha < \omega_0 \) we get \( s_1, s_2 \) complex and conjugate. Our \( v(t) \) will consist of a sinusoid (at angular frequency \( \omega_d = \sqrt{\omega_0^2 - \alpha^2} \)) multiplied by an exponentially decaying envelope \( e^{-\alpha t} \).

\[
v(t) = (B_1 \cos \omega_d t + B_2 \sin \omega_d t) e^{-\alpha t}
\]

Having found \( s_1 \) and \( s_2 \), we find \( A_1 \) and \( A_2 \) (or \( D_1, D_2 \) or \( B_1, B_2 \)) by making our \( v(t) \) satisfy initial conditions (i.e. at \( t = 0^+ \)): \( v(t = 0^+) \) and \( \frac{dv}{dt} (t = 0^+) \). (See later problems.)

We are now ready to answer part (a):

The characteristic equation roots are \( s_1, s_2 = \frac{-\alpha \pm \sqrt{\omega_0^2 - \omega_d^2}}{2} \)

\[
\alpha = \frac{1}{2RC} = \frac{1}{2 \cdot \frac{kF}{\mu} \cdot \frac{8nF}{80 \mu}} = \frac{1}{8 \ \text{rad/s}} = 12.5 \ \text{kHz}\n\]
\[ w_0^2 = \frac{1}{LC} = \frac{1}{1.25 \text{ H} \times 8 \text{ nF}} = \frac{1}{10 \text{ n}} = 100 \text{ rad/s}^2 \]
\[ \sqrt{\frac{2}{w_0^2} - \omega^2} = \sqrt{(12.5k)^2 - (10k)^2} \text{ rad/s} \]
\[ = 2.5k \sqrt{5^2 - 4^2} \text{ rad/s} \]
\[ = 2.5k \cdot 3 = 7.5k \text{ rad/s} \]

\[ s_1, s_2 = -12.5k \pm 7.5k \text{ rad/s} \]

\[ s_1 = -5k \text{ rad/s}, \quad s_2 = -20k \text{ rad/s} \]

b) \( \alpha > w_0 \) (real roots \( s_1 \neq s_2 \)) so is overdamped.

c) \( w_d = \sqrt{w_0^2 - \alpha^2} = 6k \text{ rad/s} \) desired

\[ w_0^2 = \frac{1}{LC} \text{ unaffected by } R \text{ is still } (10k)^2 \text{ rad/s}^2 \]

\[ w_d^2 = w_0^2 - \alpha^2 \text{ from squaring both sides of } w_d = \sqrt{w_0^2 - \alpha^2} \]

\[ \text{or } \alpha^2 = w_0^2 - w_d^2 \quad \text{or } \alpha = \sqrt{w_0^2 - w_d^2} = \sqrt{(10k)^2 - (6k)^2} \text{ rad/s} \]

\[ \text{or } \alpha = 8k \text{ rad/s} \]

Now \( \alpha = \frac{1}{2RC} \Rightarrow R = \frac{1}{2 \cdot 8k \cdot 8n} = 1 \text{ rad/s} = 1M \text{ rad/s} \]

\[ R = 7.8125 \text{ k} \Omega \]

d) \( s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - w_0^2} = -8k \pm j6k \text{ rad/s} \) (since \( w_d = 6k \text{ rad/s} \))

e) Critically damped when \( s_1 = s_2 \Rightarrow \alpha = w_0 = 10k \text{ rad/s} \)

\[ R = \frac{1}{2 \cdot 10k \cdot 8n} = \frac{1}{160} \text{ rad/s} = 1M = 6.25 \text{ k} \Omega \]