1. The general integral that gives the beam solid angle is equation (2-142)

\[
\Omega_A = \int \int |F(\theta, \phi)|^2 d\Omega = \int_0^{2 \pi} \int_0^\pi |F(\theta, \phi)|^2 \sin \theta d\theta d\phi
\]

There is no \( \phi \) dependence to the given antenna pattern, so the \( \phi \) integral can be done immediately, giving a factor of \( 2\pi \). The integral is then split up, separating out the three non-zero regions.

\[
\Omega_A = 2\pi \left[ \int_0^{\pi/6} \sin \theta d\theta + \int_{\pi/3}^{2\pi/3} \sin \theta d\theta + \int_{\pi/2}^{5\pi/6} \sin \theta d\theta \right]
\]

Do the integration. The integral of sine is minus cosine. We can absorb the sign by swapping the limits.

\[
\Omega_A = 2\pi \left[ \cos 0 - \cos \frac{\pi}{6} + \frac{1}{9} \left( \cos \frac{\pi}{3} - \cos \frac{2\pi}{3} \right) + \frac{1}{4} \left( \cos \frac{5\pi}{6} - \cos \pi \right) \right]
\]

Evaluating the cosine function.

\[
\Omega_A = 2\pi \left[ \left( 1 - \frac{\sqrt{3}}{2} \right) + \frac{1}{9} \left( \frac{1}{2} + \frac{1}{2} \right) + \frac{1}{4} \left( -\frac{\sqrt{3}}{2} + 1 \right) \right] = 2\pi \left[ \frac{5}{4} \left( 1 - \frac{\sqrt{3}}{2} \right) + \frac{1}{9} \right]
\]

Plugging in the numbers, we have

\[
\Omega_A = 1.750
\]

The directivity is then

\[
D = \frac{4\pi}{\Omega_A} = 7.179
\]
2. We can start from the expressions in the solution of 6.3-8. For the parasitic element to function as the best possible director, we need to maximize the array factor magnitude in the forward direction

\[ \text{maximize} |AF(0)| = \text{maximize} |I_0 + I_1e^{j\beta d}| \]

To reach this maximum, the two terms must have the same phase, modulo 2\(\pi\)

\[ \arg(I_1e^{j\beta d}) = \arg(I_0) + 2\pi n \]

Evaluating the argument function

\[ \arg(I_1) + \arg(e^{j\beta d}) = \arg(I_0) + 2\pi n \]
\[ \arg(\frac{j}{2}) + \arg(e^{j\beta d}) = \arg(1) + 2\pi n \]
\[ \frac{\pi}{2} + \beta d = 2\pi n \]

Solve for \(d\) and substitute the expression for \(\beta\)

\[ d = \left( n - \frac{1}{4} \right) \lambda \]

The shortest (positive) distance is

\[ d = \frac{3}{4} \lambda \]

but integral additions of \(\lambda\) also work. Similarly, for the parasitic element to function as the best possible reflector, we need to maximize the array factor magnitude in the backward direction

\[ \text{maximize} |AF(\pi)| = \text{maximize} |I_0 + I_1e^{-j\beta d}| \]

Following from above, we find

\[ \frac{\pi}{2} - \beta d = 2\pi n \]

Solving for \(d\)

\[ d = \left( \frac{1}{4} - n \right) \lambda \]

and the shortest distance is

\[ d = \frac{1}{4} \lambda \]
3. The source points we want lie on a circle of radius \( a \) in the \( x\)-\( y \) plane, spaced apart by an angle of \( \pi/3 \). Noting that the first point has an angle of \( \pi/6 \), we subtract 1/2 from the index variable.

\[
\mathbf{r}'_n = a \left\{ \cos \left( \left( n - \frac{1}{2} \right) \frac{\pi}{3} \right) \hat{x} + \sin \left( \left( n - \frac{1}{2} \right) \frac{\pi}{3} \right) \hat{y} \right\}
\]

If this is not obvious, consider Euler’s formula

\[
e^{j\psi} = \cos \psi + j \sin \psi
\]

The radial unit vector has the usual expansion in cartesian coordinates

\[
\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}
\]

We just need the dot product to compute the far-field approximate distances

\[
R_n = r - r'_n \cdot \hat{r} = r - a \sin \theta \left\{ \cos \left( \left( n - \frac{1}{2} \right) \frac{\pi}{3} \right) \cos \phi + \sin \left( \left( n - \frac{1}{2} \right) \frac{\pi}{3} \right) \sin \phi \right\}
\]

The expression for the individual distances is also acceptable. Evaluating the trigonometric functions we have

\[
\begin{align*}
R_1 &= r - a \sin \theta \left\{ \frac{\sqrt{3}}{2} \cos \phi + \frac{1}{2} \sin \phi \right\} \\
R_2 &= r - a \sin \theta \{ \sin \phi \} \\
R_3 &= r - a \sin \theta \left\{ -\frac{\sqrt{3}}{2} \cos \phi + \frac{1}{2} \sin \phi \right\} \\
R_4 &= r - a \sin \theta \left\{ -\frac{\sqrt{3}}{2} \cos \phi - \frac{1}{2} \sin \phi \right\} \\
R_5 &= r - a \sin \theta \{ -\sin \phi \} \\
R_6 &= r - a \sin \theta \left\{ \frac{\sqrt{3}}{2} \cos \phi - \frac{1}{2} \sin \phi \right\}
\end{align*}
\]