First let me remind you how we get to equation (2-173). A small element of power dissipation on the wire is

\[ dP_o = \frac{1}{2} dR |I(z)|^2 \]

where the resistance, \( dR \), of this small element is given by the surface resistance times the element length (in the current direction) divided by the cross-sectional length, which is the circumference of the wire

\[ dR = R_s \frac{dz}{2\pi a} \]

The total dissipated power is the sum (integral) of the power dissipated in all these differential elements, and also equal to the power dissipated in the equivalent resistance, \( R_o \)

\[ P_o = \int_{-\Delta z/2}^{\Delta z/2} dP_o = \int_{-\Delta z/2}^{\Delta z/2} \frac{1}{2} R_s \frac{dz}{2\pi a} |I(z)|^2 = \frac{1}{2} R_o |I_A|^2 \]

Solving for \( R_o \)

\[ R_o = R_s \frac{L_{\text{eff}}}{2\pi a} \quad L_{\text{eff}} \equiv \int_{-\Delta z/2}^{\Delta z/2} \frac{|I(z)|^2}{I_A} dz \]

where I have defined and effective dissipation length for the wire antenna.

(a) Now for a uniform current distribution

\[ I(z) = I_A \quad |z| \leq \frac{\Delta z}{2} \]

the effective length is

\[ L_{\text{eff}} = \int_{-\Delta z/2}^{\Delta z/2} dz = \Delta z \]

and thus the equivalent resistance for dissipation is

\[ R_o = R_s \frac{\Delta z}{2\pi a} \]
(b) For the triangular current distribution of the short dipole

\[ I(z) = I_A \left( 1 - \frac{2|z|}{\Delta z} \right) \quad |z| \leq \frac{\Delta z}{2} \]

the effective length is

\[ L_{\text{eff}} = \int_{-\Delta z/2}^{\Delta z/2} \left( 1 - \frac{2z}{\Delta z} \right)^2 \, dz = 2 \int_0^{\Delta z/2} \left( 1 - \frac{2z}{\Delta z} \right)^2 \, dz \]

One can do a variable substitution

\[ \zeta = \frac{2z}{\Delta z} \quad d\zeta = \frac{2}{\Delta z} \, dz \]

Making the integral easily solvable and dimensionless (which is usually a good idea)

\[ L_{\text{eff}} = \Delta z \int_0^1 (1 - \zeta)^2 \, d\zeta = \frac{1}{3} \Delta z \]

and thus the equivalent resistance for dissipation is

\[ R_o = R_s \frac{\Delta z}{6\pi a} \]
2.7-4 (a) The radiation resistance for the short dipole is

\[ R_r = \frac{1}{24\pi} \eta (\beta \Delta z)^2 \]

and the loss resistance of the short dipole is

\[ R_o = R_r \frac{\Delta z}{6\pi a} \]

where the radiation surface resistance is

\[ R_s = \sqrt{\frac{\omega \mu \beta}{2\sigma}} = \sqrt{\frac{\beta \eta}{2\sigma}} \]

so that the resistance ratio is

\[ r \equiv \frac{R_r}{R_o} = \frac{1}{24\pi} \eta (\beta \Delta z)^2 \frac{\beta \eta \Delta z}{\sqrt{2\sigma} 6\pi a} \]

simplifying we have

\[ r = \sqrt{\frac{\sigma \eta}{8 \beta}} \beta^2 a \Delta z \]

We note that the square root and the factor after it are both dimensionless, so that our ratio is dimensionless, as it should be. Also, note that the conductivity, \( \sigma \), is a property of the antenna metal, and the intrinsic impedance, \( \eta \), is a property of the surrounding medium.

(b) First note that the radiation efficiency of a dipole and the equivalent monopole are the same. This can be seen as follows. If we assume the same current distribution on the upper half of the dipole and the monopole, the monopole will radiate half the power of the dipole (just the power into the upper half space.) Also the monopole will dissipate just half the power of the dipole (equal to the power the dipole dissipates on just its upper half.) So, we can use example 2-4 to verify our dipole formula. Just remember the dipole’s corresponding length is twice that of the monopole. Plugging in the numbers

\[ f = 1 \text{ MHz} \]
\[ c = 2.9979 \times 10^8 \text{ m/s} \]
\[ \beta = \frac{2\pi f}{c} = 0.0210 \text{ 1/m} \]
\[ \sigma = 2 \times 10^6 \text{ S/m} \text{  (silicon steel in the table, but this is the value used in (2-180))} \]
\[ \eta = 376.73 \Omega \]
\[ a = 1.5875 \text{ mm} \]
\[ \Delta z = 2h = 2 \times 0.787 \text{ m} \]
We obtain

\[
\begin{align*}
  r &= 0.0736 \quad \text{and} \quad e_r = \frac{R_r}{R_r + R_o} = \frac{r}{r + 1} = 0.0685
\end{align*}
\]

just as in example 2-4.

(c) Our design formula should have the length expressed in terms of the efficiency. First solve the expression from part (a) for the length.

\[
\Delta z = \frac{1}{\beta^2 a} \sqrt{\frac{8\beta}{\sigma \eta}} r
\]

Then express the ratio, \( r \), in terms of the efficiency

\[
r = \frac{e_r}{1 - e_r}
\]

and substitute to obtain

\[
\Delta z = \frac{1}{\beta^2 a} \sqrt{\frac{8\beta}{\sigma \eta}} \frac{e_r}{1 - e_r}
\]

Plugging in the numbers

\[
\begin{align*}
  f &= 100 \text{ MHz} \\
  c &= 2.9979 \times 10^8 \text{ m/s} \\
  \beta &= \frac{2\pi f}{c} = 2.10 \text{ 1/m} \\
  \sigma &= 5.8 \times 10^7 \text{ S/m} \\
  \eta &= 376.73 \text{  } \Omega \\
  a &= \frac{1}{2} \times 1.024 \text{. mm} \\
  e_r &= 0.90
\end{align*}
\]

We obtain

\[
\Delta z = 111 \text{ mm}
\]
This problem is an application of the continuity equation

\[ \nabla \cdot \mathbf{J} = -j \omega \rho \]

The current density

\[ \mathbf{J} = \hat{z} J_0 \sin \left[ \beta \left( \frac{\Delta z}{2} - |z| \right) \right] \quad |z| \leq \frac{\Delta z}{2} \]

Since the current density only has a z-component (and only depends on z), the divergence is just the z-derivative of the z-component. Do the derivatives for z less than and greater than zero separately. For z less than zero

\[ \nabla \cdot \mathbf{J} = \frac{d}{dz} J_0 \sin \left[ \beta \left( \frac{\Delta z}{2} + z \right) \right] = J_0 \beta \cos \left[ \beta \left( \frac{\Delta z}{2} + z \right) \right] \]

For z greater than zero

\[ \nabla \cdot \mathbf{J} = \frac{d}{dz} J_0 \sin \left[ \beta \left( \frac{\Delta z}{2} - z \right) \right] = -J_0 \beta \cos \left[ \beta \left( \frac{\Delta z}{2} - z \right) \right] \]

Combining these expressions

\[ \nabla \cdot \mathbf{J} = -\text{sign}(z) J_0 \beta \cos \left[ \beta \left( \frac{\Delta z}{2} - |z| \right) \right] \]

Then solving the continuity equation for \( \rho \)

\[ \rho = \frac{j}{\omega} \nabla \cdot \mathbf{J} = -j \text{sign}(z) \frac{\beta}{\omega} J_0 \cos \left[ \beta \left( \frac{\Delta z}{2} - |z| \right) \right] \]

and finally we have

\[ \rho = -j \text{sign}(z) \frac{J_0}{c} \cos \left[ \beta \left( \frac{\Delta z}{2} - |z| \right) \right] \]

Note that \( J_0/c \) has the correct units of charge per unit volume.
The charge is fairly uniform (if the dipole is short) and has opposite sign on the two dipole halves.

For the half-wave dipole, the charge is continuous at the feed point.
3.2-6 Use the usual approximation for the vector potential in the far-field

\[ A = \mu \frac{e^{-jbr}}{4\pi r} \int v' \mu \frac{e^{-jbr}}{4\pi r} \int J e^{jbr^{'}} dv' \]

For our linear \( z \)-directed currents

\[ r^{'} \cdot \hat{r} = (z^{'} \hat{z}) \cdot ((\sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}) = z^{'} \cos \theta \]

Then with current centered on the origin, and spanning length \( \Delta z \)

\[ A = \mu \frac{e^{-jbr}}{4\pi r} \int_0^{\Delta z/2} I(z^{'})e^{jbc' \cos \theta} dz^{'} \]

\[ = \hat{z} \mu \frac{e^{-jbr}}{4\pi r} \left[ \int_0^{\Delta z/2} I(z^{'})e^{jbc' \cos \theta} dz^{'} + \int_{-\Delta z/2}^0 I(z^{'})e^{jbc' \cos \theta} dz^{'} \right] \]

\[ = \hat{z} \mu \frac{e^{-jbr}}{4\pi r} \left[ \int_0^{\Delta z/2} I(z^{'})e^{jbc' \cos \theta} dz^{'} - \int_{-\Delta z/2}^0 I(z^{'})e^{jbc' \cos \theta} dz^{'} \right] \]

\[ = \hat{z} \mu \frac{e^{-jbr}}{4\pi r} \left[ \int_0^{\Delta z/2} I(z^{'})e^{jbc' \cos \theta} dz^{'} + \int_0^{\Delta z/2} I(-z^{'})e^{-jbc' \cos \theta} dz^{'} \right] \]

\[ = \hat{z} \mu \frac{e^{-jbr}}{4\pi r} \left[ \int_0^{\Delta z/2} I(z^{'})e^{jbc' \cos \theta} dz^{'} + \int_0^{\Delta z/2} I(z^{'})e^{-jbc' \cos \theta} dz^{'} \right] \]

\[ = \hat{z} \mu \frac{e^{-jbr}}{4\pi r} \int_0^{\Delta z/2} I(z^{'}) \left( e^{jbc' \cos \theta} + e^{-jbc' \cos \theta} \right) dz^{'} \]

\[ = \hat{z} \mu \frac{e^{-jbr}}{2\pi r} \int_0^{\Delta z/2} I(z^{'}) \cos (bc' \cos \theta) dz^{'} \]

where I have done the following steps to simplify the integral: split up the integral into two parts, swapped the limits in the second integral, changed integration variable of the second integral from \( z^{'} \) to \( -z^{'} \), used the fact that the current distribution is symmetric in \( z^{'} \), combined the integrals, and used the definition of cosine in terms of exponentials. Now our current distribution is

\[ I(z^{'}) = I_m \sin \left( \frac{\lambda}{4} \right) \]

\[ |z^{'}| \leq \frac{\lambda}{4} \]

Plugging into the above, dropping the absolute value (since \( z^{'} \) is positive over the integration range), and using the fact that \( \Delta z \) is half the wavelength

\[ A = \hat{z} \mu I_m \frac{e^{-jbr}}{2\pi r} \frac{\lambda^{1/4}}{4} \sin \left( \frac{\lambda}{4} - z^{'} \right) \cos (bc' \cos \theta) dz^{'} \]
Do the variable substitution

\[ u = \beta z' \]

Then we have

\[ A = \hat{\mathbf{z}} \frac{\mu I_m}{2\pi} \frac{e^{-jbr}}{\beta r} f(\theta) \]

where the integral is now

\[ f(\theta) \equiv \int_0^{\pi/2} \sin\left(\frac{\pi}{2} - u\right) \cos(u \cos \theta) du = \int_0^{\pi/2} \cos(u) \cos(u \cos \theta) du = \frac{\cos\left(\frac{\pi}{2} \cos \theta\right)}{\sin^2 \theta} \]

according to Mathematica. Using equation (2-106) for \( z \)-directed vector potential

\[
E = j \frac{\omega \mu I_m}{2\pi} \frac{e^{-jbr}}{\beta r} \sin \theta \frac{\cos\left(\frac{\pi}{2} \cos \theta\right)}{\sin^3 \theta} \hat{\theta}
\]
3.3-2 (a) In the upper half space, the quarter-wave monopole has the same radiation pattern as the half-wave dipole.

(b) The quarter-wave monopole radiates half the power of the corresponding half-wave dipole, but has the same radiation intensity (in the upper half space).

\[
D_{\text{mono}} = \frac{U_{\text{mono}}}{P_{\text{mono}} / 4\pi} = \frac{U_{\text{dip}}}{(P_{\text{dip}} / 2) / 4\pi} = 2 \frac{U_{\text{dip}}}{P_{\text{dip}} / 4\pi} = 2D_{\text{dip}} = 2 \times 1.64 = 3.28
\]

(c) Since the electric field of the quarter-wave monopole and the corresponding half-wave dipole is the same and the dipole electric field is symmetric around the gap center

\[
V_{A,\text{dip}} = -\int_{-\delta z}^{\delta z} E_{\text{dip}} \cdot dl = -2 \int_{0}^{\delta z} E_{\text{dip}} \cdot dl = -2 \int_{0}^{\delta z} E_{\text{mono}} \cdot dl = 2V_{A,\text{mono}}
\]

However, the corresponding antennas have the same current distribution, including the feed terminal current, \( I_A \).

\[
Z_{A,\text{mono}} = \frac{V_{A,\text{mono}}}{I_{A,\text{mono}}} = \frac{V_{A,\text{dip}}/2}{I_{A,\text{dip}}} = \frac{1}{2} Z_{A,\text{dip}} = \frac{1}{2} R_{A,\text{dip}}
\]

where, for the last equality, we used the fact that the reactance of a tuned half-wave dipole is zero. We can’t say anything more specific since the wire radius and conductivity were not given.