Assignment 3 Solutions

3.4-1 This is a straightforward application of the source-to-observation-point radial distance approximation.

\[ R = r - \hat{r} \cdot r' \]

We apply this approximation, just at the centers of the four loop segments. (The total loop is analyzed as a superposition of four ideal dipoles, rather than integrating around the loop.)

\[ r'_1 = -\frac{l}{2} \hat{y} \quad r'_2 = \frac{l}{2} \hat{x} \quad r'_3 = \frac{l}{2} \hat{y} \quad r'_4 = -\frac{l}{2} \hat{x} \]

The unit vector in the origin-to-observation-point direction, is the usual equation (C-4)

\[ \hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \]

Taking the dot products we obtain equation (3-44)

<table>
<thead>
<tr>
<th>( R_1 )</th>
<th>( R_2 )</th>
<th>( R_3 )</th>
<th>( R_4 )</th>
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<tbody>
<tr>
<td>( r + \frac{l}{2} \sin \theta \sin \phi )</td>
<td>( r - \frac{l}{2} \sin \theta \cos \phi )</td>
<td>( r - \frac{l}{2} \sin \theta \sin \phi )</td>
<td>( r + \frac{l}{2} \sin \theta \cos \phi )</td>
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</table>
3.5-4 (a) We know the element/pattern factor for the half-wave dipole already, equation (3-4)

\[ F(\theta) = g(\theta) f(\theta) = \frac{\cos \left( \frac{\pi}{2} \cos \theta \right)}{\sin \theta} \]

So we just need the array factor. The general expression for the array factor is

\[ AF = \sum_n I_n e^{j\beta \hat{r} \cdot \hat{r}'_n} \]

Here we have \( \lambda \) spacing on the z-axis, and equal amplitude and phase driving, so we set the coefficients to unity (and handle the normalization later).

\[ \hat{r}'_1 = -\frac{\lambda}{2} \hat{z} \quad \hat{r}'_2 = \frac{\lambda}{2} \hat{z} \]

\[ I_1 = I_2 = 1 \]

The relevant dot products are

\[ \beta \hat{r} \cdot \hat{r}'_1 = \beta \left( -\frac{\lambda}{2} \hat{z} \right) \left( \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \right) = -\beta \frac{\lambda}{2} \cos \theta = -\frac{2\pi \lambda}{\lambda} \cos \theta = -\pi \cos \theta \]

\[ \beta \hat{r} \cdot \hat{r}'_2 = \beta \left( \frac{\lambda}{2} \hat{z} \right) \left( \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \right) = \beta \frac{\lambda}{2} \cos \theta = \frac{2\pi \lambda}{\lambda} \cos \theta = \pi \cos \theta \]

So that the array factor is

\[ AF = e^{-j\pi \cos \theta} + e^{j\pi \cos \theta} = 2 \cos(\pi \cos \theta) \]

The total antenna array pattern is then

\[
\begin{align*}
F(\theta) &= g(\theta) f(\theta) AF(\theta) = \frac{\cos \left( \frac{\pi}{2} \cos \theta \right) \cos(\pi \cos \theta)}{\sin \theta} \\
\end{align*}
\]

Plotting reveals that unity normalization is realized by dropping the 2.
(b) The total array pattern is shown in blue, the half-wave dipole pattern in red. The 3D pattern is shown below.
5.1-3 The far-zone electric field is equation (5-6)

\[ E_\theta = \frac{j\omega \mu e^{-jbr}}{4\pi r} I_0 L \sin \theta \text{sinc } u \]

where for broadside

\[ u = \frac{\beta L}{2} \left( \cos \theta - \cos \frac{\pi}{2} \right) = \frac{\beta L}{2} \cos \theta \]

so that

\[ E_\theta = \frac{j\omega \mu e^{-jbr}}{4\pi r} I_0 L \sin \theta \text{sinc } \left( \frac{\beta L}{2} \cos \theta \right) \]

For a short line source

\[ L \ll \lambda \text{ or } \frac{L}{\lambda} \ll 1 \text{ or } \beta L \ll 1 \]

For small argument, the sinc function is approximated by the first term in its Taylor series, which is unity.

\[ \text{sinc } u = 1 - \frac{u^2}{6} + \frac{u^4}{120} + \ldots \]

So, for the short line source

\[ E_\theta \approx \frac{j\omega \mu e^{-jbr}}{4\pi r} I_0 L \sin \theta \]

which is the same as the ideal dipole far-zone expression (2.74a) with the identification of \( L \) with \( \Delta z \).
5.2-7 (a) Broadside means that the current distribution is in phase

\[ \theta_0 = \frac{\pi}{2} \quad \beta_0 = -\beta \cos \frac{\pi}{2} = 0 \]

so that the current distribution is

\[ I(z) = I_0 \left(1 - \frac{2}{L}|z|\right) \quad -\frac{L}{2} < z < \frac{L}{2} \]

For a far-field result with a \( z \)-oriented line current, we can start from equation (2-103)

\[
A = \hat{z} \mu \frac{e^{-jbr}}{4\pi r} \int I(z) e^{jbr' \cos \theta} dz' = \hat{z} \mu \frac{e^{-jbr}}{4\pi r} \int I_0 \left(1 - \frac{2}{L}|z'|\right) e^{jbr' \cos \theta} dz' \\
= \hat{z} \mu I_0 \frac{e^{-jbr}}{4\pi r} \left[ \int_{-L/2}^{0} \left(1 + \frac{2}{L} z'\right) e^{jbr' \cos \theta} dz' + \int_{0}^{L/2} \left(1 - \frac{2}{L} z'\right) e^{jbr' \cos \theta} dz' \right] \\
= \hat{z} \mu I_0 \frac{e^{-jbr}}{4\pi r} \left[ \int_{-L/2}^{0} \left(1 + \frac{2}{L} z'\right) e^{jbr' \cos \theta} dz' + \int_{0}^{L/2} \left(1 - \frac{2}{L} z'\right) e^{jbr' \cos \theta} dz' \right] \\
= \hat{z} \mu I_0 \frac{e^{-jbr}}{4\pi r} \left[ \int_{0}^{L/2} \left(1 - \frac{2}{L} z'\right) e^{jbr' \cos \theta} dz' + \int_{0}^{L/2} \left(1 - \frac{2}{L} z'\right) e^{jbr' \cos \theta} dz' \right] \\
= \hat{z} \mu I_0 \frac{e^{-jbr}}{4\pi r} \left[ \int_{0}^{L/2} \frac{L}{2} \cos(\beta z' \cos \theta) dz' \right] \\
= \hat{z} \mu I_0 \frac{e^{-jbr}}{4\pi r} \left[ \int_{0}^{u} \left(1 - \frac{v}{u}\right) \cos(v) dv \right]
\]

Here we have: split the integral to eliminate the absolute value, swapped the limits (in the first integral), changed integration variable by a minus sign (in the first integral), recombined the integrals, and finally changed the integration variable as

\[ v = \beta z' \cos \theta \quad \frac{z'}{\beta \cos \theta} = v \quad \frac{dz'}{\beta \cos \theta} = dv \]

and used the usual definition for \( u \), in the broadside case

\[ u = \frac{\beta L}{2} \cos \theta \]

Evaluating the integral and using the half-angle formula

\[
\int_{0}^{u} \left(1 - \frac{v}{u}\right) \cos(v) dv = \frac{1}{u} \int_{0}^{u} \left(1 - \frac{v}{u}\right) \cos(v) dv = \frac{1}{u} \left[ 1 - \cos u \right] = \frac{2 \sin^2 (u/2)}{u^2} = \frac{2 \sin^2 (u/2)}{4 (u/2)^2} = \frac{1}{2} \frac{1}{\sin^2 (u/2)}
\]
We find the vector potential is

\[ A = \hat{z} \mu L \frac{e^{-j\beta r}}{4\pi r} \frac{1}{2} \text{sinc}^2 \left( \frac{u}{2} \right) \]

and from equation (2-106)

\[ E = j\omega \sin \theta A \hat{\theta} \]

we obtain

\[ E_{\theta} = \frac{IL}{8\pi} j\omega \frac{e^{-j\beta r}}{r} \sin \theta \text{sinc}^2 \left( \frac{\beta L}{4\cos \theta} \right) \]

(b) As in the previous problem

\[ L \ll \lambda \quad \Rightarrow \quad \frac{L}{\lambda} \ll 1 \quad \Rightarrow \quad \beta L \ll 1 \quad \Rightarrow \quad u \ll 1 \quad \Rightarrow \quad \text{sinc} u \approx 1 \]

So that the electric field is approximately

\[ E_{\theta} \approx \frac{IL}{8\pi} j\omega \frac{e^{-j\beta r}}{r} \sin \theta \]

(c) The far-zone electric field expression for the ideal dipole is

\[ E_{\theta} = \frac{IL}{4\pi} j\omega \frac{e^{-j\beta r}}{r} \sin \theta \]

which is exactly twice as large as the short limit of the triangular-current dipole. This supports the idea that, for electrically short line currents, the far-field amplitude scales as the integral of the current distribution. In this case the triangular current distribution has half the area of the rectangular (uniform) current distribution.
To compare velocities take their ratio and use the expression of velocity in terms of frequency and phase constant
\[
\frac{v}{c} = \frac{\omega / \beta_0}{\omega / \beta} = \frac{\beta}{\beta_0}
\]

For a wave traveling in the positive \( z \) direction, we find the phase as a function of \( z \)
\[
\psi(z) = e^{-j\beta z} \quad \Rightarrow \quad \phi = -\beta z \quad \Rightarrow \quad \beta = -\frac{\Delta\phi}{\Delta z}
\]

(a) Find the effective phase constant for the wire array
\[
\beta_0 = \frac{-80^\circ \frac{\pi}{180^\circ}}{\frac{\lambda}{4}} = \frac{8}{9} \frac{2\pi}{9} = \frac{8}{9} \beta
\]

Then the find the velocity ratio
\[
\frac{v}{c} = \frac{9}{8} \quad \Rightarrow \quad \text{fast wave}
\]

(b) Find the effective phase constant for the wire array
\[
\beta_0 = \frac{-100^\circ \frac{\pi}{180^\circ}}{\frac{\lambda}{4}} = \frac{10}{9} \frac{2\pi}{9} = \frac{10}{9} \beta
\]

Then the find the velocity ratio
\[
\frac{v}{c} = \frac{9}{10} \quad \Rightarrow \quad \text{slow wave}
\]