The power delivered to any of the resistors in the circuit is given by

\[ P_R = \frac{1}{2} \Re \left( V_R I^* \right) = \frac{1}{2} R |I|^2 \quad \text{where} \quad I = \frac{V_g}{R_g + jX_g + R_o + R_r + jX_A} \]

(a) For reactive tuning we have

\[ X_g = -X_A \quad \Rightarrow \quad I = \frac{V_g}{R_g + R_o + R_r} \quad \Rightarrow \quad P_R = \frac{1}{2} \frac{R}{(R_g + R_o + R_r)^2} |V_g|^2 \]

noting that the problem text means \( X_g \) when it says \( X_L \). Then the power radiated, the power dissipated in the source resistance, and the power dissipated in the antenna “ohmic” resistance are, respectively

\[
\begin{align*}
P &= \frac{1}{2} \frac{R}{(R_g + R_o + R_r)^2} |V_g|^2 \\
P_g &= \frac{1}{2} \frac{R_g}{(R_g + R_o + R_r)^2} |V_g|^2 \\
P_o &= \frac{1}{2} \frac{R_o}{(R_g + R_o + R_r)^2} |V_g|^2 
\end{align*}
\]

(b) In addition, if we have real load matching

\[ R_g = R_r + R_o \]

the above expression simplify to

\[
\begin{align*}
P &= \frac{1}{8} \frac{R}{(R_o + R_r)^2} |V_g|^2 \\
P_g &= \frac{1}{8} \frac{1}{R_g} |V_s|^2 \\
P_o &= \frac{1}{8} \frac{1}{(R_o + R_r)^2} |V_s|^2 
\end{align*}
\]

The total power supplied is given by

\[ P_{tot} = \frac{1}{2} \frac{R_g + R_o + R_r}{(R_g + R_o + R_r)^2} |V_s|^2 = \frac{1}{2} \frac{2R_g}{(2R_g)^2} |V_s|^2 = \frac{1}{4} \frac{1}{R_g} |V_s|^2 \]

so that the fraction of supplied power that is dissipated in the source resistor is

\[ \frac{P_g}{P_{tot}} = \frac{1}{2} \]
The surface resistance is given by (2-171)

\[ f = 27 \text{ MHz} \]
\[ \mu = 4\pi \times 10^{-7} \text{ H/m} \]
\[ \sigma = 3.5 \times 10^{-7} \text{ S/m} \quad \text{(aluminum from B.1)} \]
\[ R_s = \sqrt{\frac{\omega \mu}{2\sigma}} = \sqrt{\frac{\pi f \mu}{\sigma}} = 0.00175 \text{ \Omega} \]

We have an expression for the “Ohmic” resistance for a dipole with triangular current distribution, (2-175)

\[ \Delta z = \frac{\lambda}{2} = \frac{c}{2f} = 5.55 \text{ m} \]
\[ d = 0.00635 \text{ m} \]
\[ R_o = \frac{\Delta z}{2\pi a} \frac{R_s}{3} = \frac{\Delta z}{\pi d} \frac{R_s}{3} = 0.162 \text{ \Omega} \]

Then the efficiency is

\[
\begin{align*}
R_r &= 70 \text{ \Omega} \\
\varepsilon_r &= \frac{R_r}{R_r + R_o} = 0.998
\end{align*}
\]

Very good efficiency. Probably very few of you know what a citizen’s band (CB) radio is. Before the internet and mobile phones they were somewhat culturally significant, especially to truckers, hicks and nerds.
2.8-4 The incident RHCP wave, propagating in the positive \( z \)-direction, has electric field at the plane of reflection is given by
\[
E_i = E_0 (\hat{x} - j\hat{y})
\]
The electric field \( y \)-component lags the \( x \)-component, so the field rotates counter-clockwise.

The reflected wave propagates in the opposite direction. We represent its electric field in the following (right-handed) coordinate system
\[
\hat{x'} = -\hat{x} \quad \hat{y'} = \hat{y} \quad \hat{z'} = -\hat{z}
\]

Thus the reflected wave propagates in the positive \( z' \)-direction. Its electric field at the plane of reflection is given by
\[
E_r = E_{rx}\hat{x'} + E_{ry}\hat{y'}
\]

The total electric field at the reflection plane must be zero, since the reflection plane is a perfect electric conductor
\[
E = E_i + E_r = E_0 (\hat{x} - j\hat{y}) + E_{rx}\hat{x'} + E_{ry}\hat{y'} = E_0 (\hat{x} - j\hat{y}) - E_{rx}\hat{x} + E_{ry}\hat{y} = 0
\]

The \( x \)- and \( y \)-components of this equation yield
\[
E_{rx} = E_0 \quad E_{ry} = jE_0
\]

So that the reflected wave electric field is
\[
E_r = E_0 (\hat{x'} + j\hat{y'})
\]

Here the electric field \( y' \)-component leads the \( x' \)-component, so the field rotates clockwise in the primed coordinate system, which is LHCP.

Alternatively, you could show that the incident and reflected wave rotate the same direction in the same coordinate system, but since the waves propagate in opposite directions, if one is RHCP the other is LHCP.
This problem is sort of trivial. Use the formula for a non-fringing, parallel plate capacitor

\[ C = \frac{\varepsilon A}{d} \]

where \( A \) is the plate area, \( d \) is the plate separation and \( \varepsilon \) is the permittivity between the plates. From figure 3-3

\[ A = \pi \Delta r^2 \quad d = \Delta z \]

and the antenna is assumed to be in air

\[ \varepsilon = \varepsilon_0 \]

Plug in to find

\[ C = \frac{\varepsilon_0 \pi \Delta r^2}{\Delta z} \]
The “ohmic” resistance for cylindrical wire antennas, oriented along the z-axis, is given by (2-173)

\[ R_0 = \frac{1}{|I_A|^2} \frac{R_s}{2\pi a} \int_{-\lambda/4}^{\lambda/4} |I(z)|^2 dz \]

The half-wave dipole current distribution is given by

\[ I(z) = I_m \sin \left( \beta \left( \frac{\lambda}{4} - |z| \right) \right) \quad |z| \leq \frac{\lambda}{4} \]

The antenna terminal current is

\[ I_A = I(0) = I_m \sin \left( \frac{\beta \lambda}{4} \right) = I_m \sin \left( \frac{2\pi \lambda}{\lambda} \right) = I_m \sin \left( \frac{\pi}{2} \right) = I_m \]

The appropriate integration limits are given by the non-zero range of the current distribution, i.e. \( \pm \lambda/4 \). Plugging in we have

\[ R_0 = \frac{R_s}{2\pi a} \int_{-\lambda/4}^{\lambda/4} \sin^2 \left( \beta \left( \frac{\lambda}{4} - |z| \right) \right) dz \]

Since the integrand is symmetric, we can perform the integration over just the positive \( z \) range, and drop the absolute value on \( z \)

\[ R_0 = 2 \frac{R_s}{2\pi a} \int_0^{\lambda/4} \sin^2 \left( \beta \left( \frac{\lambda}{4} - z \right) \right) dz \]

Make the substitution

\[ \theta = \beta z \]
\[ d\theta = \beta dz \]

\[ R_0 = 2 \frac{R_s}{2\pi a} \frac{\pi/2}{\beta} \int_0^{\pi/2} \sin^2 \left( \frac{\pi}{2} - \theta \right) d\theta = \frac{R_s}{2\pi a} \frac{2}{\beta} \int_0^{\pi/2} \cos^2 \theta d\theta = \frac{R_s}{2\pi a} \frac{2 \pi}{\beta} \]

where the integral is easily done since we all know that the average value of sine or cosine squared is one-half. Finally converting \( \beta \) back to \( \lambda \), we have

\[ R_0 = \frac{R_s}{2\pi a} \frac{\lambda}{4} \]
3.3-1 There are a couple ways to go on this. You could orient the z-axis parallel to the dipoles or perpendicular to the ground plane, and thus matching the symmetry of either the sources or the boundary condition, but not both. I am going with former, since we have solutions for z-oriented dipoles. We have from (2-73) with the somewhat complicated radial dependence represented by arbitrary function

\[ E_i = I_i \left[ f(r_i) \sin \theta \hat{\theta} + g(r_i) \cos \theta \hat{r} \right] \]

\[ E_2 = I_2 \left[ f(r_2) \sin \theta \hat{\theta} + g(r_2) \cos \theta \hat{r} \right] \]

Note that we have a different spherical coordinate system centered on the location of each dipole, with different basis vectors. The corresponding cartesian basis vectors are, of course, the same for both. From the following diagram

we can find the relationships between the projections of the radial vectors

\[ r_2 \cdot \hat{x} = r_1 \cdot \hat{x} \quad r_2 \cdot \hat{y} = -r_1 \cdot \hat{y} \quad r_2 \cdot \hat{z} = r_1 \cdot \hat{z} \]

which, in terms of the radial magnitudes and angles, is

\[ r_2 \sin \theta \cos \phi = r_1 \sin \theta \cos \phi \quad r_2 \sin \theta \sin \phi = -r_1 \sin \theta \sin \phi \quad r_2 \cos \theta = r_1 \cos \theta \]

Since these coordinates are independent, one can choose specific values to show that

\[ r_2 = r_1 \quad \theta_2 = \theta_1 \quad \phi_2 = -\phi_1 \]

Now, combining the two fields and using

\[ I_2 = -I_1 \]

we find

\[ E = E_1 + E_2 = I_i \left[ f(r_1) \sin \theta \hat{\theta} - f(r_2) \sin \theta \hat{\theta} + g(r_1) \cos \theta \hat{r} - g(r_2) \cos \theta \hat{r} \right] \]
Using the equality of the radial coordinates and the azimuthal angles for points on the ground plane, this simplifies to

\[ E = I_1 \left[ f(r_1) \sin \theta_1 \left( \hat{\theta}_1 - \hat{\theta}_2 \right) + g(r_1) \cos \theta_1 (\hat{r}_1 - \hat{r}_2) \right] \]

From (C-4) and (C-5) we have

\[
\hat{r}_1 - \hat{r}_2 = \hat{x} \sin \theta_1 \cos \phi_1 + \hat{y} \sin \theta_1 \sin \phi_1 + \hat{z} \cos \theta_1 - \left( \hat{x} \sin \theta_2 \cos \phi_2 + \hat{y} \sin \theta_2 \sin \phi_2 + \hat{z} \cos \theta_2 \right) \\
\hat{\theta}_1 - \hat{\theta}_2 = \hat{x} \cos \theta_1 \cos \phi_1 + \hat{y} \cos \theta_1 \sin \phi_1 - \hat{z} \sin \theta_1 - \left( \hat{x} \cos \theta_2 \cos \phi_2 + \hat{y} \cos \theta_2 \sin \phi_2 - \hat{z} \sin \theta_2 \right)
\]

Simplifying using the relations above, for points on the boundary, only the terms with a \( \sin \phi \) do not cancel.

\[
\hat{r}_1 - \hat{r}_2 = \hat{y} 2 \sin \theta_1 \sin \phi_1 \\
\hat{\theta}_1 - \hat{\theta}_2 = \hat{y} 2 \cos \theta_1 \sin \phi_1
\]

Substituting into the electric field expression we obtain

\[ E = 2I_1 \sin \theta_1 \cos \theta_1 \sin \phi_1 \left[ f(r_1) + g(r_1) \right] \hat{y} \]

which is oriented along the \( y \)-axis for all points on the ground plane, and thus normal to the ground plane, satisfying the boundary condition.