3.4-2 Start from the fields given by (3-49) and (3-50)

\[ E = \eta \beta^2 S \frac{I e^{-j\beta r}}{4\pi r} \sin \theta \phi \]

\[ H = -\beta^2 S \frac{I e^{-j\beta r}}{4\pi r} \sin \theta \hat{\phi} \]

You can then find the power radiated using both the electric and magnetic field (2-128) or just the electric field (2-130), but note that (2-128) has a typo. One can find the typo by calculating the Poynting vector for transverse electric and magnetic fields.

\[ S = \frac{1}{2} E \times H^* = \frac{1}{2} \left( E_\phi \hat{\theta} + E_\theta \hat{\phi} \right) \times \left( H_\phi \hat{\theta} + H_\theta \hat{\phi} \right)^* = \frac{1}{2} \left( E_\theta H_\phi^* \hat{\theta} \times \hat{\phi} + E_\phi H_\theta^* \hat{\theta} \times \hat{\phi} \right) = \frac{1}{2} \left( E_\theta H_\phi^* - E_\phi H_\theta^* \right) \hat{r} \]

The real (average) power is then given by

\[ P = \text{Re} \int \int_{\text{sphere at } r} S \cdot ds = \frac{1}{2} \text{Re} \int \int_0^{2\pi} \left( E_\theta H_\phi^* - E_\phi H_\theta^* \right) r^2 \sin \theta d\theta d\phi \]

where we have recalled that the directed area on a sphere is given by

\[ ds = (r d\theta)(r \sin \theta d\phi) \hat{r} \]

Plugging the small loop fields into this expression we find

\[ P = \frac{1}{2} \text{Re} \int_0^{2\pi} \int_0^\pi \left[ -\eta \beta^2 S \frac{I e^{-j\beta r}}{4\pi r} \sin \left( -\beta^2 S \frac{I e^{-j\beta r}}{4\pi r} \sin \theta \right) \right] r^2 \sin \theta d\theta d\phi \]

Simplifying and noting that the integrand is real valued we find

\[ P = \frac{1}{32\pi^2} \eta |I|^2 \beta^4 S^2 \int_0^{2\pi} \int_0^\pi \sin^2 \theta d\theta d\phi \]

The \( \phi \) integral is \( 2\pi \), as always for \( \phi \) independent integrands, and the \( \theta \) integral is

\[ \int_0^\pi \sin^2 \theta d\theta = \frac{4}{3} \]
The power is then

\[ P = \frac{1}{12\pi} \eta |I|^2 \beta^4 S^2 \]

To obtain equation (3-52) - which is inferior to the equation above - we employ the commonly used approximation for the free-space impedance in Ohms

\[ \eta \approx 120\pi \quad [\Omega] \]

Then the power is given by unit the specific equation

\[ P \approx 10|I|^2 (\beta^2 S)^2 \quad [I \text{ in Amps gives } P \text{ in Watts}] \]
3.5-2(a) The total pattern is the product of the element pattern, $g_a$, and the normalized array factor, $f$.

$$F(\theta, \phi) = g_a(\theta, \phi) f(\theta, \phi)$$

Note that the product of normalized functions is not necessarily normalized. The element pattern, for the short dipole, is given in Table 3-2. If you were curious and worked out the exact antenna pattern for the triangular current distribution you may have been surprised that the result is different than Table 3-2. In addition to having a triangular current distribution, the short dipole is assumed to be short. If you expand the exact result in a series you will see that the leading term is the table result. See appendix below.

$$g_a(\theta) = \sin \theta$$

In general, the array factor is

$$AF = \sum_i \alpha_i e^{j\beta_i r_i}$$

In this problem we have

$$\alpha_1 = \alpha_2 = 1$$

$$r_1' = -\frac{d}{2} \mathbf{\hat{z}}$$

$$r_2' = \frac{d}{2} \mathbf{\hat{z}}$$

Using (C-4) we find

$$\mathbf{\hat{z}} \cdot \mathbf{\hat{r}} = \cos \theta$$

So that the array factor is

$$AF = e^{-j\frac{2d}{2} \cos \theta} + e^{j\frac{2d}{2} \cos \theta} = 2 \cos \left( \beta \frac{d}{2} \cos \theta \right)$$

If we normalize this we have

$$f(\theta) = \cos \left( \beta \frac{d}{2} \cos \theta \right)$$

The total pattern is

$$F(\theta) = \sin \theta \cos \left( \beta \frac{d}{2} \cos \theta \right) = \sin \theta \cos (0.4 \pi \cos \theta) \quad \text{when} \quad d = 0.4 \lambda$$

which is normalized since the the two factors have their maximums at the same angle.
4.2-3 (a) Equating the last parts of equations (4-8) and (4-10) we have

\[
\frac{1}{2} \left( \frac{|V_A|^2}{(R_A + R_L)^2 + (X_A + X_L)^2} \right) R_L = \frac{1}{2} \frac{|E|^2}{\eta} A_{em}
\]

You should also use the definition of effective length, equation (4-1), so that the expression for effective area won’t refer to the particular excitation level

\[
\frac{1}{2} \left( \frac{|V_A|^2}{(R_A + R_L)^2 + (X_A + X_L)^2} \right) R_L = \frac{1}{2} \frac{|V_A|^2}{h} A_{em}
\]

Solve for the effective area

\[
A_{em} = h^2 \frac{\eta R_L}{(R_A + R_L)^2 + (X_A + X_L)^2}
\]

which has the correct dimension. Strictly speaking, this is not the absolute maximum effective area, since this expression includes the effects of impedance mismatch.

(b) When we have a conjugate matched load

\[
R_L = R_A \quad \text{and} \quad X_L = -X_A
\]

and the above result reduces to

\[
A_{em} = h^2 \frac{\eta R_A}{(R_A + R_A)^2 + (-X_L + X_L)^2} = \frac{1}{4} h^2 \frac{\eta}{R_A}
\]

which is the same as equation (4-12)
4-3.2 One thing you need for this problem is the solid angle of the interior of a cone with specified cone angle, $\theta_c$. As always, solid angles are the same as area on the unit sphere. Integrating the areas of the annular regions that compose the

$$\Omega = \int_{0}^{\theta_c} 2\pi \sin \theta \, d\theta = -2\pi \left[ \cos \theta \right]_0^{\theta_c} = -2\pi \left[ \cos \theta_c - 1 \right] = 4\pi \sin^2 \left( \frac{\theta_c}{2} \right)$$

In class I said you could use the area of the base of the cone as an approximation to the area of the spherical cap.

$$\Omega \approx \pi \sin^2 \theta_c$$

You can see that for small cone angle, these expressions are equivalent. Next, given the directivity of the beam we can find the solid angle of the beam, using the usual expression, (2-144)

$$D = \frac{4\pi}{\Omega_A}$$

We are given the formula for the antenna temperature that is dominated by small hotspot, equation (4-19). Plugging in the above equations

$$T_A = \frac{\Omega}{\Omega_A} T_s = \frac{4\pi \sin^2 \left( \frac{\theta_c}{2} \right)}{4\pi / D} T_s$$

which simplifies to

$$T_A = D \sin^2 \left( \frac{\theta_c}{2} \right) T_s$$

Using the provided values

$$\begin{array}{c}
\theta_c = \frac{0.5^o}{2} = 0.25 \frac{\pi}{180} \\
D = 40\text{dB} = 10^4 \\
T_s = 6000\text{K} \\
T_A \approx 286\text{K}
\end{array}$$

If you had approximated the sun as a square, you would get 364K, which is an unnecessarily bad approximation.
4.4-3 (a,b,c) Equation (4-23) allows to relate the maximum effective area to the directivity.

\[ D = \frac{4\pi}{\lambda^2} A_{em} \]

The directivities of an isotropic radiator, a short dipole, and a half-wave dipole are given in Table 3-2. The corresponding maximum effective areas are

\[ D = 1, \ 1.5, \ 1.64 \]

\[ A_{em} = \frac{D}{4\pi} \lambda^2 = 0.080\lambda^2, \ 0.119\lambda^2, \ 0.131\lambda^2 \]

For the half-wave dipole at 100MHz

\[ c = 300 \text{ m/}\mu\text{S} \]
\[ f = 100 \text{ MHz} \]
\[ D = 1.64 \]
\[ A_{em} = \frac{D}{4\pi} \left( \frac{c}{f} \right)^2 = 1.17 \text{ m}^2 \]

Whereas the relevant physical area is just the geometric cross-section

\[ d = 0.003 \text{ m} \]
\[ A = dL = \frac{d\lambda}{2} = \frac{dc}{2f} = 0.0045 \text{ m}^2 \]

which is about 300 times smaller. The electromagnetic cross-section of small resonant objects can be of order \( \lambda^2 \) even if their physical cross-section is much smaller.
4.5-5 You can use equation (4.57) or the form of the Friis transmission equation given in class

\[ \frac{P_r}{P_t} = \left( \frac{\lambda}{4\pi R} \right)^2 G_t G_r q_t q_r P R_0 \left( \frac{f_0}{f} \right)^{m-2} \]

where the first six factors comprise the exact relation of unobstructed antennas in infinite free-space, and the last two factors represent a simple empirical fading model, which may apply in a specific, non-ideal environment, with judicious choice of the model parameters: \( m, n, R_0, f_0 \).

\[ c = 0.3 \text{ km/µS} \quad f = 1900 \text{ MHz} \quad R = 8 \text{ km} \]
\[ G_t = 16.8 \text{ dB} \quad G_r = -1 \text{ dB} \]
\[ q_t = 1 \quad q_r = 0.75 \quad p = 0.5 \]
\[ n = 4 \quad R_0 = 1 \text{ km} \quad m = 2 \quad (f_0 \text{ is not relevant}) \]
\[ P_t = 20 \text{ W} \]
\[ P_r = 11.0 \text{ pW} = -79.5 \text{ dBm} \]

I think there may be an issue with adding the receiver and antenna noise temperatures when there is an impedance mismatch. (I will ignore that for now, and address it later.)

\[ T_{\text{sys}} = T_A + T_r \]

The system noise power is

\[ P_{N\text{sys}} = k T_{\text{sys}} \Delta f \]

If we take our power transmitted and received to be the carrier power, the carrier to noise ratio is

\[ \text{CNR} = \frac{P_r}{P_{N\text{sys}}} = \frac{P_r}{k (T_A + T_r) \Delta f} \]

Plugging in the numbers, we have

\[ T_A = 290 \text{ K} \quad T_r = 1500 \text{ K} \]
\[ k = 1.381 \times 10^{-23} \text{ J/K} \]
\[ \Delta f = 1.5 \text{ MHz} \]
\[ \text{CNR} = 296 = 24.7 \text{ dB} \]

Which is quite usable in most cases.
3.5-2(a) Appendix. Exact antenna pattern for triangular current distribution.

\[ A = \hat{\mathbf{z}} \mu \frac{e^{-j\beta r}}{4\pi r} \int I(z') e^{j\beta' c \cos \theta} dz' \]

\[ I(z) = I_A \left( 1 - \frac{2|z|}{\Delta z} \right) |z| \leq \frac{\Delta z}{2} \]

\[ A = \hat{\mathbf{z}} \mu \frac{e^{-j\beta r}}{4\pi r} \left[ \int_{-\Delta z/2}^{0} I_A \left( 1 + \frac{2z'}{\Delta z} \right) e^{j\beta c \cos \theta} dz' + \int_{0}^{\Delta z/2} I_A \left( 1 - \frac{2z'}{\Delta z} \right) e^{j\beta c \cos \theta} dz' \right] \]

\[ u = \beta z' \quad du = \beta dz' \quad l = \frac{\beta \Delta z}{2} \]

\[ A = \hat{\mathbf{z}} \mu I_A \frac{e^{-j\beta r}}{\beta} \frac{l}{4\pi r \sin \theta} F(\theta) \quad \text{where} \quad F(\theta) \equiv \frac{\sin \theta}{l} \left[ \int_{-\frac{l}{2}}^{0} \left( 1 + \frac{u}{l} \right) e^{ju \cos \theta} du + \int_{0}^{\frac{l}{2}} \left( 1 - \frac{u}{l} \right) e^{ju \cos \theta} du \right] \]

\[ E = j\omega \sin \theta \hat{A} \hat{\theta} \]

\[ E = j\omega \mu I_A l \frac{e^{-j\beta r}}{\beta} \frac{l}{4\pi r} F(\theta) \hat{\theta} \]

\[ E = j\eta I_A l \frac{e^{-j\beta r}}{4\pi r} F(\theta) \hat{\theta} \]

\[ F(\theta) = \frac{\sin \theta}{l} \left[ \int_{0}^{\frac{l}{2}} \left( 1 - \frac{u}{l} \right) e^{-ju \cos \theta} du + \int_{0}^{\frac{l}{2}} \left( 1 - \frac{u}{l} \right) e^{ju \cos \theta} du \right] \]

\[ F(\theta) = \frac{\sin \theta}{l} \int_{0}^{\frac{l}{2}} \left( 1 - \frac{u}{l} \right) 2 \cos (u \cos \theta) du \]

\[ F(\theta) = \frac{2}{l^2} \left[ 1 - \cos (l \cos \theta) \right] \sec \theta \tan \theta \]

\[ F(\theta) = \sin \theta - \frac{1}{12} (\cos^2 \theta \sin \theta) l^2 + O(l^4) \quad l \ll 1 \]