1. a. (5 points)

Calculate $v_1$. 

b. (5 points)

Calculate $i_1$. 

ans: a) 40 V  
b) 2 A
**sol'n: (a)** The 100 V source is directly across $20 \, k\Omega \parallel 30 \, k\Omega$ in series with $8 \, k\Omega$. Thus, the rest of the circuit is irrelevant in the calculation of $v_1$.

![电路图](image)

$20 \, k\Omega \parallel 30 \, k\Omega = 10 \, k\Omega \cdot 2 || 3 = 10 \, k\Omega \cdot \frac{2 \cdot 3}{2 + 3} = 10 \, k\Omega \cdot \frac{6}{5} = 12 \, k\Omega$

Now we have a voltage divider.

$v_1 = 100 \, V \cdot \frac{8 \, k\Omega}{12 \, k\Omega + 8 \, k\Omega} = 40V$

**sol'n: (b)** The 10 A source current is in series with $40 \, k\Omega \parallel 10 \, k\Omega$. Thus, all of the 10 A must flow through the $40 \, k\Omega \parallel 10 \, k\Omega$, and the rest of the circuit is irrelevant in the calculation of $i_1$. We use the current divider formula, and we may ignore the $100 \, k\Omega$ resistor.

![电路图](image)

$i_1 = 10 \, A \cdot \frac{10 \, k\Omega}{10 \, k\Omega + 40 \, k\Omega} = 2 \, A$
2. (30 points)

Derive an expression for \( i_3 \). The expression must not contain more than the circuit parameters \( V_a, V_b, i_a, R_1, R_2, \) and \( R_3 \).

\[ i_3 = \frac{V_a R_2 + V_b (R_1 + R_2) - i_a R_1 R_2}{R_1 R_2 + R_3 (R_1 + R_2)} \]

**sol'n:** Using passive sign convention, label voltage drop and current measurement polarities.

Use Kirchhoff's laws:

- sum v drops around loop = 0
- sum i out of node = 0

v drops for loop on left, using Ohm's law for \( v_1 \):

\[ V_a - i_1 R_1 - i_2 R_2 = 0 \text{ V} \]

Middle loop would include current source, so use slightly larger loop with \( R_2 \) on left and \( V_b \) on right:

\[ i_2 R_2 - i_3 R_3 + V_b = 0 \text{ V} \]

Now sum currents out of top node (that consists of the two top nodes connected by a wire).

Note: We are always allowed to combine nodes connected by wires.

\[ -i_1 + i_2 + i_a + i_3 = 0 \text{ A} \]

We now have three equations in three unknowns. We solve for \( i_3 \). Use the second equation to eliminate \( i_2 \):
\( i_2 = \frac{i_3 R_3 - V_b}{R_2} \)

Use the first equation to eliminate \( i_1 \):
\[
i_1 = \frac{V_a - i_2 R_2}{R_1} = \frac{1}{R_1} \left[ V_a - \left( \frac{i_3 R_3 - V_b}{R_2} \right) \right] = \frac{1}{R_1} \left( V_a + V_b - i_3 R_3 \right)
\]

Substitute for \( i_1 \) and \( i_2 \) in the third equation:
\[
-\frac{1}{R_1} \left( V_a + V_b - i_3 R_3 \right) + \frac{i_3 R_3 - V_b}{R_2} + i_a + i_3 = 0 \text{ A}
\]

Solve for \( i_3 \):
\[
-\frac{1}{R_1} \left( V_a + V_b \right) - \frac{V_b}{R_2} + i_a + \frac{1}{R_1} i_3 R_3 + \frac{i_3 R_3}{R_2} + i_3 = 0 \text{ A}
\]
\[
i_3 \left( \frac{R_3}{R_1} + \frac{R_3}{R_2} + 1 \right) = \frac{1}{R_1} \left( V_a + V_b \right) + \frac{V_b}{R_2} - i_a
\]

Multiply both sides by \( R_1 R_2 \) to clear fractions:
\[
i_3 \left( R_3 R_2 + R_3 R_1 + R_1 R_2 \right) = R_2 \left( V_a + V_b \right) + R_1 V_b - i_a R_1 R_2
\]

or
\[
i_3 = \frac{V_a R_2 + V_b \left( R_1 + R_2 \right) - i_a R_1 R_2}{R_1 R_2 + R_3 \left( R_1 + R_2 \right)}
\]

Now for consistency checks to verify our answer. (Optional)

1) Consider \( i_a = 0 \), \( V_b = 0 \), and \( R_3 = 0 \):

Since \( R_2 \) is bypassed by a short, no current flows in \( R_2 \). Therefore, we can remove \( R_2 \) without changing \( i_3 \):
2) Consider \( i_a = 0 \) (open circuit) and \( R_2 = \infty \) (open circuit):

Removing \( R_2 \) and \( i_a \) leaves total voltage \( V_a + V_b \) across \( R_1 + R_3 \) in outside loop.

Therefore, we have

\[
i_3 = \frac{V_a + V_b}{R_1 + R_3}
\]

For our formula, we use the following identities:

\[
\lim_{R_2 \to \infty} \frac{R_2}{R_1 R_2 + R_4 (R_1 + R_2)} = \frac{1}{R_1 + R_3} \quad \text{and} \quad \lim_{R_2 \to \infty} \frac{R_1 + R_2}{R_1 R_2 + R_3 (R_1 + R_2)} = \frac{1}{R_1 + R_3}
\]

Making these substitutions in our formula gives

\[
i_3 = \frac{V_a + V_b}{R_1 + R_3}
\]

(3) Consider \( V_a = 0, i_a = 0 \):

\[
i_3 = \frac{V_b}{R_1 || R_2 + R_3}
\]

Our formula gives \( i_3 = \frac{V_b R_1 + R_2}{R_1 R_2 + R_3 (R_1 + R_2)} \) or \( i_3 = \frac{V_b}{R_1 || R_2 + R_3} \). 

\( \checkmark \)
(4) Consider \( V_a = 0, V_b = 0 \):

\[ V_a = 0 \quad \Rightarrow \quad i_3 = -\frac{i_a R_1 \parallel R_2}{R_1 \parallel R_2 + R_3} \]

By current divider formula, we have

\[ i_3 = -\frac{i_a R_1 \parallel R_2}{R_1 \parallel R_2 + R_3} \]

Our formula gives

\[ i_3 = -\frac{i_a R_1 \parallel R_2}{R_1 \parallel R_2 + R_3} \]

(5) Consider \( i_a = 0, V_b = 0 \).

\[ V_a = 0 \quad \Rightarrow \quad i_3 = -\frac{i_a R_1 \parallel R_2}{R_1 \parallel R_2 + R_3} \]

\[ i_3 = -\frac{i_a R_1 \parallel R_2}{R_1 \parallel R_2 + R_3} \]

Our formula gives

\[ i_3 = -\frac{i_a R_1 \parallel R_2}{R_1 \parallel R_2 + R_3} \]

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Our formula gives

\[ i_3 = -\frac{i_a R_1 \parallel R_2}{R_1 \parallel R_2 + R_3} \]

\[ i_3 = -\frac{i_a R_1 \parallel R_2}{R_1 \parallel R_2 + R_3} \]
3.  (30 points)

a. Derive an expression for $i_2$. The expression must not contain more than the circuit parameters $\alpha$, $V_a$, $i_a$, $R_1$, and $R_2$.

\[ R_1 + \alpha v_1 - R_2 v_1 - R_2 i_a - V_a = 0 \]

b. Make at least one consistency check (other than a units check) on your expression. Explain the consistency check clearly.

**ans:** a) $i_3 = \frac{V_a R_2 + V_b (R_1 + R_2) - i_a R_1 R_2}{R_1 R_2 + R_3 (R_1 + R_2)}$

b) Many possible answers. See solution below.

**sol'n:** (a) Use Kirchhoff's laws to write several equations. Then eliminate unwanted variables.

We sum currents out of top-center node:

\[-i_1 - i_a + i_2 = 0\]

Note that summing currents out of bottom-center node does not give us anything new. By Ohm's law, we also have

\[ i_1 = \frac{v_1}{R_1} \]

Now, we sum voltages around a loop. We choose the outer loop because the inner loops have a current source with unknown voltage drop.

\[ \alpha v_1 - v_1 + V_a - v_2 = 0 \quad \text{or} \quad (\alpha - 1)v_1 + V_a - v_2 = 0 \]

By Ohm's law, we also have

\[ v_2 = i_2 R_2 \]

After the Ohm's law substitutions, we have two equations, and we may eliminate $v_1$.  

Use the simpler equation first:

\[
\frac{v_1}{R_1} + i_a - i_2 = 0 \quad \text{or} \quad v_1 = (i_2 - i_a)
\]

Substitute for \(v_1\) in the second equation:

\[
(\alpha - 1)R_1(i_2 - i_a) + V_a - i_2R_2 = 0
\]

After some algebra, we get

\[
i_2 = \frac{(1-\alpha)i_aR_1 + V_a}{(1-\alpha)R_1 + R_2} \quad \text{units consistent} \quad \checkmark
\]

(b) There are many possible consistency checks.

1) \(i_a = 0\) and \(R_1 = 0\). Then \(v_1 = 0\), \(\alpha v_1 = 0\), and sum v's around outer loop gives \(i_2 = V_a/R_2\). Our formula also gives \(V_a/R_2\). \(\checkmark\)

2) Consider \(R_1 = 0\) and \(R_2 = 0\). As in (1), \(v_1 = 0\) and \(\alpha v_1 = 0\). Since \(R_2 = 0\) we also end up with a short across \(V_a\):

![Diagram](image)

We expect \(i_2 \to \infty\) for short across \(V_a\)

Our formula gives

\[
\lim_{R_2 \to 0} \frac{V_a}{R_2} = \infty \quad \text{for} \quad i_2 \quad \text{from (1)} \quad \checkmark
\]

3) Consider \(i_a = 0\), \(\alpha = 0\):

![Diagram](image)

Our formula gives

\[
i_2 = \frac{V_a}{R_1 + R_2} \quad \text{by Ohm's law} \quad \checkmark
\]
4) Consider $R_1 \to \infty$ (open circuit)

Clearly $i_a = i_2$ since the same current flows through elements in series.

Our formula gives:

$$i_2 = \lim_{R_1 \to \infty} \frac{(1-\alpha)i_a R_1 + V_a}{(1-\alpha)R_1 + R_2} = \lim_{R_1 \to \infty} \frac{(1-\alpha)i_a R_1}{(1-\alpha)R_1} = i_a$$

5) Consider $R_2 \to \infty$ (open circuit): We have $i_2 = 0$ since no current flows through the open circuit. Our formula gives:

$$i_2 = \lim_{R_2 \to \infty} \frac{(1-\alpha)i_a R_1 + V_a}{(1-\alpha)R_1 + R_2} = \text{const} \to 0$$

Many more consistency checks are possible.
4. (30 points)

The op amp operates in the linear mode. Using an appropriate model of the op amp, derive an expression for \(v_o\) in terms of not more than \(i_s, R_1, R_2,\) and \(R_3.\)

\[
\text{ans: } v_o = i_s \left( R_2 + R_3 + \frac{R_2 R_3}{R_1} \right)
\]

\[
\text{sol'n: Using passive sign convention, label voltage drop and current measurement polarities.}
\]

Assume an ideal op-amp since we are in linear mode and we are solving for DC conditions.

Use the standard procedure to find \(v_o:\)

1) Calculate \(v_p:\) We use Kirchhoff's law for summation of currents out of the node next to the + terminal of the op-amp. Since \(i_p = 0,\) (no current flows into the + terminal), we conclude that \(i_2 = i_p\)

By Ohm's law, it follows that \(v_p = i_s R_2\)

2) Assume \(v_n = v_p:\) Because of negative feedback, the op-amp output voltage, \(v_o,\) reaches an equilibrium level that results in \(v_n \approx v_p.\) Now we have the following model circuit:
Note that we model the – terminal of the op-amp as a voltage source with zero current flowing into it. We add this constraint on the current with the understanding that \( v_o \) must have a value that results in \( i_n = 0 \).

3) Calculate \( i'_s \), the current flowing toward the – terminal from the left: Now we observe that voltage source \( v_n \) separates the two sides of the circuit. What happens on the right side of the circuit will not affect \( i'_s \).

We use a summation of node currents at the node above \( R_1 \) to calculate the value of \( i'_s \):

\[
i'_s + i_1 + i_s = 0 \quad \text{or} \quad i'_s = -(i_1 + i_s)
\]

But we can also calculate \( i_1 \) because we have \( v_n \) across \( R_1 \):

\[
i_1 = \frac{v_n}{R_1} = \frac{i_s R_2}{R_1}
\]

4) Calculate \( i_3 \) using only the right side of the circuit: What happens on the left side of the \( v_n \) source will not affect this calculation of \( i_3 \). We use a voltage loop around the right side of the circuit.

\[
v_n - i_3 R_3 - v_o = 0 \quad \text{or} \quad i_3 = \frac{v_n - v_o}{R_3} \quad \text{or} \quad i_3 = \frac{i_s R_2 - v_o}{R_3}
\]

5) Since \( i_n = 0 \), set \( i'_s = i_3 \): Now we are saying that the two sides of the circuit DO affect each other in that \( v_o \) must have a value that causes \( i_n = 0 \).

\[
-(\frac{i_s R_2}{R_1} + i_s) = \frac{i_s R_2 - v_o}{R_3}
\]

6) Solve for \( v_o \):

\[
v_o = i_s (R_2 + R_3 + \frac{R_2 R_3}{R_1}) \quad \text{units consistent} \quad \checkmark
\]

**Consistency checks:**

1) By linearity and fact that we have one source, \( i_s \), we expect \( v_o \) to be directly proportional to \( i_s \). Agrees with the formula. \( \checkmark \)
2) Consider \( R_2 = 0 \): Then \( v_p = 0 \) and \( v_n = v_p = 0 \). Therefore, \( i_1 = 0 \) and \( i_3 = -i_s \). Then, \( v_o = -i_3R_3 = i_sR_3 \). Agrees with the formula. ✓

3) Consider \( R_1 = \infty \) (open circuit): \( v_n = v_p = i_sR_2 \), \( i_3 = -i_s \), and \( v_o = v_n - i_3R_3 = i_sR_2 + i_sR_3 \). Agrees with formula. ✓

4) Consider \( R_3 = 0 \): Then \( v_o = v_n = v_p = i_sR_2 \). Agrees with the formula. ✓