1. Choose values of $L$ and $C$ that will produce a $\omega_0$ of $2 \times 10^4$ rad/s and a $Q$ of 2.

b. Calculate $b$, $\omega_{c1}$, and $\omega_{c2}$.

**Answer:**

a. $L = 3.2$ mH, $C = 80$ nF

b. $\omega = 31.4$ k rad/s, $\omega_{c1} = 49.1$ k rad/s, $\omega_{c2} = 80.5$ k rad/s

**Solution:**

a. This is a band reject filter. See Text, p. 732.

Note: The reason this is a band reject filter is that

$$\omega_0 = \frac{1}{LC} \quad \text{(Eq. 14.46)}$$

and

$$Q = \frac{L}{R^2C} \quad \text{(Eq. 14.50)}$$

From the text, we have $\omega_0 = \sqrt{\frac{1}{LC}} \quad \text{(Eq. 14.46)}$, and $Q = \sqrt{\frac{L}{R^2C}} \quad \text{(Eq. 14.50)}$

$$\omega_0 Q = \sqrt{\frac{1}{LC} \cdot \frac{L}{R^2C}} = \frac{1}{RC} \quad \text{or} \quad C = \frac{1}{R \omega_0 Q}$$
1(a) Continued

Thus,

\[ C = \frac{1}{100 \cdot 2 \cdot 10^4 / \text{S} \cdot 2} = \frac{1 \text{F}}{4 \cdot 10^6} = \frac{1000}{4} \text{nF} = 79.6 \text{nF} \]

\[ Q = \sqrt{\frac{L}{R^2 C}} \quad Q^2 R^2 C = L = 4 \cdot 100^2 \cdot 80 = 320 \cdot 10 \text{ knH} \]

or

\[ L = 3200 \text{ mH} = 3.2 \text{ mH} \]

b. From the text:

\[ \omega C_1 = \frac{\omega_0}{2} + \sqrt{\frac{\omega_0^2}{4} + \omega_0^2} \quad \text{(Eq. 14.51)} \quad \omega C_2 = \frac{\omega_0}{2} + \sqrt{\frac{\omega_0^2}{4} + \omega_0^2} \quad \text{(Eq. 14.54)} \]

and

\[ \omega C_1 = \frac{\omega_0}{2Q} + \sqrt{1 + \frac{1}{2Q}} \quad \text{(Eq. 14.53)} \]

\[ \omega C_2 = \frac{\omega_0}{2Q} + \sqrt{1 + \frac{1}{2Q}} \quad \text{(Eq. 14.54)} \]

By equating the terms before the radicals in these equations, we see that

\[ \frac{\omega_0}{2Q} = \omega \]  
Thus, \[ \omega = \frac{\omega_0}{Q} \]  
(Also, see p. 735 of text.)

\[ \omega = \frac{2 \omega_0}{Q} \cdot 10^4 = \omega_0 \cdot 10^4 \text{ rad/s} = 31.4 \text{ k rad/s} \]

Now we compute \( \omega C_1, \omega C_2 \):

\[ \omega C_1 = \frac{\omega_0}{2} + \sqrt{\frac{\omega_0^2}{4} + \omega_0^2} = \frac{\omega_0}{2} \cdot 10^4 + \sqrt{4 \cdot 10^8 + 4 \cdot 10^8} = \frac{\omega_0^2}{2} \cdot 10^8 \]

\[ \omega C_1 = \frac{\omega_0}{2} \cdot 10^4 + \sqrt{\frac{\omega_0^2}{4} \cdot 10^8 + 4 \cdot 10^8} = \frac{\omega_0^2}{2} \cdot 10^8 + \sqrt{\frac{17}{4} \omega_0^2 \cdot 10^8} \]

\[ = \frac{\omega_0^2}{2} \cdot 10^4 + \sqrt{\frac{17}{4} \omega_0^2} \cdot 10^4 = \frac{\sqrt{17} \omega_0^2}{2} \cdot 10^4 = 49.1 \text{ k rad/s} \]
1(b) Continued

\[ C_2 = \frac{p \cdot 10^4}{2} + \frac{\sqrt{17}}{2} \cdot 10^4 = \frac{\sqrt{17} + 1}{2} \cdot 10^4 = 80.5 \text{ k rad/s} \]

Verify: \[ \sqrt{C_1 \cdot C_2} = \sqrt{49.1 \text{ d} \cdot 80.5 \text{ k}} = 62.9 \text{ k} = \omega_0 \]
Choose one R, one L, or one C to go in the box to make the circuit a high-pass filter with a cutoff frequency $\omega_c = 10^5 \text{ rad/s}$. Give the value of the element you chose.

**Answer:**

$$L = 10 \text{ mH}$$

**Solution:**

$$H(j\omega) = \frac{V_o}{V_i} = \frac{z}{R + z} \quad z \text{ in box}$$

For high-pass, we want $|H(j\omega)| = 0$ for $\omega = 0$. So we want $z = 0$ at $\omega = 0$.

Choose $L$ so $z = j\omega L$ and $H(j\omega) \big|_{\omega=0} = \frac{0}{R} = 0$.

$$H(j\omega) = \frac{j\omega L}{R + j\omega L}$$

$\omega_c$ is $\omega$ where $\text{denom} = R = L$, i.e. $\text{Re}[\text{denom}] = \text{Im}[\text{denom}]$.

$$\omega_c = \frac{R}{L} \quad L = \frac{R}{\omega_c} = \frac{1 \text{ k}}{100 \text{ k}} = 10 \text{ mH}$$
Using not more than one each R, L, and C, design a circuit to go in the dashed-line box that will produce the |H| vs. ? shown above, that is:

- |H| = 0.5 at ? = 10^5 rps
- |H| = 1 at ? = 0
- |H| ~ 1 as ?

Specify values of R, L, and C, and show how they would be connected in the circuit. Note that a bandwidth is not specified, and you do not have to satisfy any more than the three requirements specified above.
Answer:

![Diagram of a circuit with a band-reject filter](image)

* Any $L \cdot C = 100$ ps acceptable (if part values are practical).

Solution:

Given the frequency response plot, we want a band-reject filter. Since $V_o$ is measured across $R_1$, we need $L$ in parallel with $C$ inside dashed box. (See Text, Fig. 14.31, p. 736.)

Note: If we denote dashed box by $z$, we have

$$V_o = V_i \cdot \frac{R_1}{R_1 + z} \quad \text{(V-divider).}$$

$$H(j\omega) \equiv \frac{V_o}{V_i} = \frac{R_1}{R_1 + z}$$

Note that if

$$z = j\omega L \cdot \frac{1}{j\omega C} = \frac{j\omega L}{j\omega L + \frac{1}{j\omega C}} = \frac{L}{C} \cdot \frac{1}{j\omega L + \frac{1}{j\omega C}}$$

then, at $\omega_o$, we have $j\omega L = -\frac{1}{j\omega C}$. So $z = \frac{L}{C} \mid_{\omega = \omega_o} = 0$. Thus, $H(j\omega) \mid_{\omega = \omega_o} = \frac{R_1}{R_1 + z} = 0$.

(Actually, we want 1/2. See below.) So, we do have a band-reject filter.

At $\omega = 0$,

$$z = \frac{L}{C} \cdot \frac{1}{j\cdot 0 \cdot L + \frac{1}{j\cdot 0 \cdot C}} = \frac{L}{C} \cdot \frac{1}{0 + 0} = 0$$

$$H(j\omega) \mid_{\omega = 0} = \frac{R_1}{|R_1 + z|} = \frac{|R_1|}{R_1} = 1$$
At $\omega$, 
\[ z = \frac{L/C}{j \cdot L + \frac{1}{j \cdot C}} = \frac{L/C}{j \cdot L + 0} = 0 \]

\[ |H(j\omega)| = \left| \frac{R_1}{R_1 + z} \right| = \left| \frac{R_1}{R_1} \right| = 1 \quad \checkmark \]

The remaining problem is to add an $R_2$ in the dashed box so that $|H(j\omega)| = \frac{1}{2}$ (not 0).

\[ \frac{1}{j \omega L} = \frac{1}{j \omega C} \quad \text{at} \quad \omega = \omega_o \]

If we put $R_2$ in series with the $L$ parallel $C$, then we would still have $z = R_2 + \frac{1}{R_1}$, at $\omega = \omega_o$. Thus, we must try something else.

If we put $R_2$ in parallel with $L$ parallel $C$, then we have $z = \frac{R_2}{R_1 + R_2}$ at $\omega = \omega_o$. This gives

\[ |H(j\omega)| = \left| \frac{R_1}{R_1 + R_2} \right| \quad \text{at} \quad \omega = \omega_o \]

We use $R_2 = R_1 = 1\Omega$ to get the required $|H(j\omega)| = \frac{1}{2}$ at $\omega = \omega_o$.

Now we must verify that we have the correct gain at $\omega = 0$ and $\omega$. For both cases, however, we have

\[ \frac{1}{j \omega L} = \frac{1}{j \omega C} = 0. \quad \text{The extra } R_2 \text{ in parallel still gives } z = 0. \]

$\omega = 1\omega_o$. Finally, we need $\omega_o = -10^5 \text{ rad/s (dip in plot)}$. But $\omega_o = \frac{1}{\sqrt{LC}}$

always. Therefore, we need

\[ \frac{1}{\omega_o^2} = \left( \frac{10^5}{10^2} \right)^2 = \frac{100}{10^2} = 100 \text{ ps}^2 \]

Any $LC = 100 \text{ ps}^2$ is acceptable, (unless $L$, $C$ is too large or small to be reasonable). For example,

$C = 1 \mu\text{F}, \quad L = 100 \mu\text{H}$
Find the coefficients $C_n$ for $n = -3, -2, -1, 0, 1, 2, 3$ of the exponential form of the Fourier series. You may either first find the $a_n$ and $b_n$ values and convert, or you may directly find the $C_n$ values. From the integral tables:

\[
\int x \sin(ax) \, dx = \frac{1}{a^2} \sin(ax) - \frac{x}{a} \cos(ax)
\]

\[
\int x \cos(ax) \, dx = \frac{1}{a^2} \cos(ax) + \frac{x}{a} \sin(ax)
\]

\[
\int \sin(bx) \, dx = \frac{e^{ax} (a \sin(bx) - b \cos(bx))}{a^2 + b^2}
\]

\[
\int \cos(bx) \, dx = \frac{e^{ax} (a \cos(bx) + b \sin(bx))}{a^2 + b^2}
\]

\[
\int e^{ax} \, dx = \frac{e^{ax} \left( x - \frac{1}{a^2} \right)}{a^2}
\]

\[
\int x \sin x \, dx = \sin x - x \cos x
\]

\[
\int x \cos x \, dx = \cos x + x \sin x
\]

Answer:

\[
C_k = \begin{cases} 
0 & \text{k even} \\
1 + e^{-4} & \text{k odd} \\
\frac{j \pi}{k} + 1 & \text{k odd}
\end{cases}
\]
4. Continued

Solution:

We have two possible approaches:

1. 

\[ C_k = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{j\omega_0 t} dt \]

2. 

\[
C_{k>0} = \frac{1}{2} (a_k - j b_k) \quad \text{where} \quad a_k = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos \omega_0 t \, dt \\
(b_{k>0} = b_k) \]

(except \( C_0 = a_0 \))

\[
(C_{k<0} = C^*_{-k}) \quad \quad b_k = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin \omega_0 t \, dt
\]

From approach (2), we have that \( C_{k \text{ even}} = 0 \) when \( a_{k \text{ even}} = 0 \) and \( b_{k \text{ even}} = 0 \). Our waveform has the half-wave shift flip property:

\[
f(t) = -f_{-T/2}^{T/2} \quad \text{with} \quad a_{k \text{ even}} = b_{k \text{ even}} = 0 \quad \text{and} \quad C_{k \text{ even}} = 0
\]

We'll use approach (1) for \( C_{k \text{ odd}} \) for the sake of illustration.

In the hope of exploiting symmetries, we'll choose one cycle to be from -4 to +4 (as opposed to, say, 0 to 8). We split our integral into two pieces for the two section of \( f(t) \), (with different defining equations):

\[
C_k = \frac{1}{8} \int_{-4}^{0} 4 e^{j(k+4)\theta} e^{j\omega_0 t} dt + \frac{1}{8} \int_{0}^{4} 4 e^{j\omega_0 t} dt
\]

Note: \( T = 8 \) and \( \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2} \)

\[
= \frac{e^{j\omega_0 t}}{2} \left[ e^{j\omega_0 t} \int_{-4}^{0} 4 e^{j(k+4)\theta} dt + \frac{1}{2} e^{j\omega_0 t} \int_{0}^{4} e^{j(k+4)\theta} dt \right]
\]

\[
= \frac{e^{j\omega_0 t}}{2} \left[ e^{j\omega_0 t} \int_{-4}^{0} 4 e^{j(k+4)\theta} dt + \frac{1}{2} e^{j\omega_0 t} \int_{0}^{4} e^{j(k+4)\theta} dt \right]
\]

\[
= \frac{e^{j\omega_0 t}}{2} \left[ e^{j\omega_0 t} \int_{-4}^{0} 4 e^{j(k+4)\theta} dt + \frac{1}{2} e^{j\omega_0 t} \int_{0}^{4} e^{j(k+4)\theta} dt \right]
\]

continued
4. (Continued)

\[
\begin{align*}
\frac{e^{-4}}{2} & \left[ \frac{1}{2} + \frac{e^{4}}{e^{jk} + 1} \right] + \frac{1}{2} \left[ \frac{1}{2} + \frac{e^{4}}{e^{jk} + 1} \right] \\
\frac{e^{-4}}{2} & \left[ \frac{1}{2} + \frac{1}{2} e^{4} \right] + \frac{1}{2} \left[ \frac{1}{2} + \frac{1}{2} e^{4} \right] \\
\end{align*}
\]

For \( k \) odd, we have \( e^{jk} = -1 \)

\[
\begin{align*}
C_k &= \frac{e^{-4}}{2} \left[ \frac{1 + e^{4}}{e^{jk} + 1} \right] + \frac{1}{2} \left[ \frac{e^{4} + 1}{e^{jk} + 1} \right] \\
&= \frac{e^{4} + 1}{2 e^{jk} + 1} + \frac{e^{4} + 1}{2 e^{jk} + 1} \\
&= \frac{e^{4} + 1}{jk + 1} \\
\end{align*}
\]

\[
C_k = \frac{1 + e^{4}}{jk + 1} \quad (k \text{ odd})
\]

In retrospect, we might have exploited symmetry and just doubled the second piece of the integral.
Write the time-domain expression of v(t) for the first through third harmonics.

**Answer:**

\[ v(t) = 1.6 \cos (300 kt - 69.4°) \]

**Solution:**

1. We turn each frequency of \( v_g(t) \) into a phasor.

2. We pass each frequency, \( k\omega_0 \), through the circuit by multiplying by the transfer function \( H(jk\omega_0) \).

3. The result is the phasor for frequency \( k\omega_0 \) in the output signal.

4. We convert the output signal phasor back to the time domain.

**Note:** We only have 1st and 3rd harmonics to worry about since we only have odd harmonics in \( v_g(t) \) and we were only asked to find the output signal's 1st through 3rd harmonics.

1. Turn each frequency of \( v_g(t) \) into a phasor:

   \[ k = 1: \quad \frac{16}{\pi} \sin (\omega_0 t) \cdot \frac{1}{\omega_0} \cdot \frac{16}{\pi} \quad \text{or} \quad \frac{16}{\pi} \cdot 90° \]

   \[ k = 3: \quad \frac{16}{3\pi} \sin (3\omega_0 t) \cdot \frac{1}{3\omega_0} \cdot \frac{16}{3\pi} \quad \text{or} \quad \frac{16}{3\pi} \cdot 90° \]

2. We pass each frequency, \( k\omega_0 \), through the circuit by multiplying by the transfer function \( H(jk\omega_0) \).

\[
H(j\omega) = \frac{j\omega + \frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} = \frac{V}{V_g} \quad \text{(V-divider)}
\]
5. (Continued)

\[ H(j \omega_o) = \frac{(j \omega_o)^2 LC + 1}{j \omega_o RC + (j \omega_o)^2 LC + 1} = \frac{(100k)^2 0.1m 1 + 1}{j100k 10k 1(100k)^2 0.1m 1 + 1} \]

\[ = \frac{100^2 \cdot 0.1m + 1}{j1k \cdot 100^2 \cdot 0.1m + 1} = 0 \]

\[ j1k = 0 \]

**Note:** In this case, the \[ \omega_o \] for the \[ v_g(t) \] Fourier series happens to be the same as the center (or resonant, or characteristic) frequency = \[ 1/\sqrt{LC} \]. This need not always be true.

\[ k = 3: \quad H(j3 \omega_o) = \frac{-3^2 + 1}{j3k - 3^2 + 1} = \frac{-8}{j3k - 8} \]

\[ = \frac{8}{8 - j3k} = \frac{1}{1 - \frac{j3k}{8}} = \frac{1}{\sqrt{73}} \cdot 20.6^\circ \]

\[ = \frac{8}{\sqrt{73}} \cdot 20.6^\circ \]

3. Input phasor: \[ H(j3 \omega_o) = \frac{16}{3} \cdot 90^\circ \cdot \frac{8}{\sqrt{73}} \cdot 20.6^\circ \]

\[ = \text{output phasor} = V_3 = \frac{16.8}{3\sqrt{73}} \cdot 69.4^\circ \]

4. Convert back to the time domain: (through the 1st 3 harmonics)

\[ v(t) = \frac{16.8}{3\pi\sqrt{73}} \cos (300 kt - 69.4^\circ) \]

\[ 3 \cdot \omega_o \]
6. Fill in the following table for the functions shown below.

![Graph of g(t) and h(t)]

**Answer:**

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<tr>
<th></th>
<th>g(t)</th>
<th>h(t)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>X</td>
<td>X</td>
</tr>
<tr>
<td>False</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>True</td>
<td>X</td>
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</tr>
<tr>
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</tr>
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<td>X</td>
</tr>
<tr>
<td>False</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

**Solution:**

**Notes:**

1. The period of g(t) is one lobe. We do not have mirror-image symmetry around T/4 and/or 3T/4.

2. But we do not have quarter-wave symmetry:
Not mirror image symmetric around $\frac{T}{4}$ or $\frac{3T}{4}$.

We also do not have half-wave shift flip, (text calls this 1/2-wave symmetry):

$$h(t) = \begin{cases} h \frac{T}{2} \end{cases}.$$  

3 All $a_k = 0$ if function is odd.

4 All $b_k = 0$ if function is even.