After being closed for a long time, switch opens at t = 0.

a. Specify the value of C that will make v(t) overdamped. C = 250 nF or 2 nF.
b. Write a time-domain expression for v(t), t > 0.
c. Calculate the time, t_m, at which the magnitude of v(t) is maximum.

Answer:

a. C = 250 nF
b. v(t) = 13.3 (e^{-0.4Mt} - e^{-1.6MT})
c. t_m = 1.16 \text{s}

Solution:

a. We will use the exponential solution.

Before t = 0, at t = 0^−, C is open circuit, L is short circuit.

Model:

\[ v_C(0^-) = 100V \cdot \frac{12.5\Omega}{25\Omega} = 50V \]
1a. (Continued)

\[ i_L(0) = \frac{100V}{25\Omega} = 4A \]

When the switch opens, the energy will decay owing to power dissipated by the series resistor \( R_1 \).

As \( t \to \infty \), \( C \) is open circuit, \( L \) is short circuit.

Model:

\[ \begin{align*}
&v_C(\infty) \quad v_A = 100V \\
&i_L(\infty) = 0A \\
&v_C(\infty) = 100V
\end{align*} \]

\[ R_1 \]

\[ i_L(\infty) \]

\[ v_C(\infty) \]

Since \( \frac{d}{dt} i_L(\cdot) = 0 \), we will not have a constant term in our solution for \( v(t) \).

(We conclude that \( \frac{d}{dt} i_L(\cdot) = 0 \) because \( i_L(\cdot) = 0 \) doesn't have any variation that would yield a nonzero derivative.)

Now we need the roots of the characteristic equation for a series RLC circuit.

From text, p. 378:

\[ s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \]

\[ s_{1,2} = \frac{-R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \]

or

\[ s_{1,2} = \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \]

\[ \frac{R}{2L} \]

\[ \frac{1}{\sqrt{LC}} \]

We want an overdamped response, (real roots, \( \frac{R^2}{4L^2} > \frac{1}{LC} \)).

\[ \frac{R}{2L} = \frac{12.5 \, \Omega}{(2) \cdot 6.25 \, \mu H} = \frac{12.5}{12.5} \, \text{M rad/s} = 1 \, \text{M rad/s} \]

Try each possible \( C \):

\[ C = 250 \, \text{nF} \]

\[ \frac{1}{\sqrt{6.25 \, \mu H \cdot 250 \, \text{nF}}} = \frac{1}{\sqrt{1562.5 \, \text{m} \cdot \Omega}} = \frac{1}{1.25} \]
1a. (Continued)

\[ \omega_0 = 0.8 \text{ M rad/s} < \omega^2 \quad \text{overdamped} \]

\[ C = 250 \text{ nF} \quad \omega_0 = \frac{1}{\sqrt{6.25 \Omega \cdot 2 \text{ nF}}} = \frac{1}{\sqrt{12.5 \text{ m} \Omega}} = 1.1118 \text{ M} \]

\[ \omega_0 = 8.9 \text{ M rad/s} > \omega^2 \quad \text{underdamped} \]

We need \( C = 250 \text{ nF} \) for overdamped solutions.

b. Now we use exponential solution and find coefficients to match initial conditions:

\[ v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \]

Now we find initial conditions (at \( t = 0^+ \)):

\[ v(t = 0^+) = v_C(0^+) + i_L(0^+) \cdot R_1 \quad VA \]

\[ = v_C(0^+) + i_L(0^+) \cdot R_1 \quad VA \]

since

\[ v_C(0^+) = v_C(0^-) \]

\[ i_L(0^+) = i_L(0^-) \]

Recall: \( v_C \) cannot change instantly, \( i_L \) cannot change instantly.

\[ v(t = 0^+) = 50V + 4A \cdot 12.5\Omega = 50 + 50V = 100V \quad 100V = 0V \]

Our second initial condition must yield a value for \( \frac{dv(t)}{dt} \Big|_{t=0^+} \).

We differentiate the equation at the top of the page:

\[ \frac{dv}{dt} = \frac{dv_C}{dt} + \frac{d}{dt} \left( i_L \cdot R_1 \right) \cdot \frac{d}{dt} VA \]

\[ \frac{dv_C}{dt} = \frac{i_C}{C} = \frac{i_L}{C} \quad \text{from} \quad i_C = C \frac{dv_C}{dt} \]

\[ \frac{d}{dt} (i_L \cdot R_1) = R_1 \frac{d}{dt} i_L = \frac{R_1}{L} v \quad \text{from} \quad v = L \frac{d}{dt} i_L \]

\[ \frac{d}{dt} VA = 0 \]
1b. (Continued)

\[
\left. \frac{dv}{dt} \right|_{t=0^+} = \frac{i_L(0^+)}{C} + \frac{R_1}{L} v(0^+) = \frac{4A}{250 \text{nF}} + \frac{12.5 \cdot 6.25}{2} \cdot 0
\]

\[
= \frac{4A}{0.25 \times 2} = 16 \text{ M v/s}
\]

We must satisfy:

\[
0 = v(0^+) = A_1 e^{s_1^0^+} + A_2 e^{s_2^0^+} = A_1 + A_2
\]

and

\[
16 \text{ M v/s} = \left. \frac{dv(t)}{dt} \right|_{t=0^+} = s_1 A_1 e^{s_1^0^+} + s_2 A_2 e^{s_2^0^+} = s_1 A_1 + s_2 A_2
\]

\[A_2 = \square A_1 \quad \text{(1st equation)}\]

\[
16 \text{ M v/s} = s_1 A_1 + s_2 (-A_1) = (s_1 - s_2) A_1
\]

\[
s_1 - s_2 = -a + a = 2 \left( a \right) - a = 0.6 \text{ M}^2
\]

\[
A_1 = \frac{16 \text{ M v/s}}{1.2 \text{ M}} = 13.3 \text{ v/s}
\]

\[
A_2 = -13.3 \text{ v/s}
\]

\[
s_1 = -1 \text{ M} + 0.6 \text{ M} = -0.4 \text{ M}
\]

\[
s_2 = -1 \text{ M} - 0.6 \text{ M} = -1.6 \text{ M}
\]

\[
v(t) = 13.3 e^{-0.4Mt} - 13.3 e^{-1.6Mt}
\]

c. \(v_m(t)\) occurs when \(\frac{dv(t)}{dt} = 0\)

\[
0 = \left. \frac{dv(t)}{dt} \right|_{t=0^+} = s_1 A_1 e^{s_1^t} + s_2 A_2 e^{s_2^t} = \left( s_1 e^{s_1 t} \square s_2 e^{s_2 t} \right) A_1
\]

Therefore, we need

\[
s_1 e^{s_1 t} \square s_2 e^{s_2 t} = 0 \quad \text{since} \quad A_1 \neq 0
\]

\[
s_1 e^{s_1 t} = s_2 e^{s_2 t}
\]
1c. (Continued)

\[
\frac{s_1}{s_2} = \frac{e^{s_2t}}{e^{s_1t}} = e^{(s_2 - s_1)t} = e^{(s_2 \boxminus s_1)t}
\]

\[
\ln \left( \frac{s_1}{s_2} \right) = (s_2 \boxminus s_1)t
\]

or

\[
t = \frac{1}{\frac{s_2}{s_1} \boxminus 1} \ln \frac{s_1}{s_2} = \frac{1}{\ln 1.2} \ln \frac{1.4}{1.6} = 1.16 \text{ ms}
\]
2.

\[ v_1(t) = a_v + \sum_{n=1}^{\infty} A_n \cos \left( n \omega_0 t + \varphi_n \right) \]

a. Determine Fourier series coefficients \( a_v, a_n, \) and \( b_n \) for \( v_i(t) \). List the numerical values of coefficients for average value and first through fifth harmonics for the Fourier series written in the form of phasors:

\[ v_1(t) = a_v + \sum_{n=1}^{\infty} A_n \cos \left( n \omega_0 t + \varphi_n \right) \]

b. Select components for the circuit above so it rejects the fundamental frequency of \( v_i(t) \) and so it has a bandwidth of 10 k rad/s.

Answer:

a. \( a_v = 0 \)

\[ a_n = \begin{cases} 
40 & \text{n even} \\
\frac{\sin \frac{\pi n}{4}}{\frac{\pi n}{4}} & \text{n odd}
\end{cases} \]

\[ b_n = 0 \quad \text{all n} \]

\[ A_1 = \frac{20\sqrt{2}}{\Box}, \quad \varphi_1 = 0^\circ \]
\[ A_2 = 0, \quad \varphi_2 = 0^\circ \]
\[ A_3 = \frac{20\sqrt{2}}{3\Box}, \quad \varphi_3 = 0^\circ \]
\[ A_4 = 0, \quad \varphi_4 = 0^\circ \]
\[ A_5 = \frac{4\sqrt{2}}{\Box}, \quad \varphi_5 = 0^\circ \]

Symmetries used: even function, half wave symmetry, quarter wave symmetry (slip flip symmetry).
b.  

Solution:

a. \( v_i(t) \) is symmetric around vertical axis \( \square v_i(t) \) even function, 

\[ b_n = 0 \] for all \( n \) (no \( \sin (n \omega t) \) terms in Fourier series)

If we shift \( v_i(t) \) one-half period and flip it upside down, we have \( v_i(t) \) again. Thus, we have what the text calls half-wave symmetry. (I prefer to call it slip-flip symmetry.)

\[ a_n = 0 \] for \( n \) even \( (b_n = 0 \) for \( n \) even, too, but we already know \( b_n = 0 \) all \( n \))

For the question of quarter wave symmetry, we look for a place where we could put \( t = 0 \), (not the location shown in the plot of \( v_i(t) \), perhaps), such that \( v_i(t) \) would be symmetric around both T/4 and 3T/4.

If we moved \( t = 0 \) to the location of T/4 in the plot, then we would have quarter wave symmetry.

\[ \square \] We may have quarter wave symmetry

The final test is to see if the actual location of \( t = 0 \) makes \( v_i(t) \) either even (so we only have \( a_n \) for \( n \) odd terms left) or odd (so we only have \( b_n \) for \( n \) odd terms left).

Since \( v_i(t) \) is an even function, we have only \( a_n \) for \( n \) odd terms left.

But we already knew this! What we also have, however, is a simpler formula for the calculation of \( a_n \), \( n \) odd:

\[ a_n = \frac{8}{T} \int_0^{T/4} v_i(t) \cos (n \omega t) dt, \quad n \text{ odd} \]

(Compare this with the original formula \( a_n = \frac{2}{T} \int_0^T v_i(t) \cos (n \omega t) dt \).)

Now we observe that

\[ v_i(t) = \begin{cases} 10 & 0 < t < T/8 \\ 0 & T/8 < t < T/4 \end{cases} \]
Thus

\[ a_n = \frac{8}{T} \int_{T/8}^{T} 10 \cos \left( n_{o} t \right) dt + \frac{T/4}{T/8} \int_{T/8}^{T} 0 \cdot \cos \left( n_{o} t \right) dt \]

or

\[ a_n = \frac{8}{T} \int_{T/8}^{T} 10 \cos \left( n_{o} t \right) dt \]

\[ = \frac{8}{T} \left. 10 \sin \left( n_{o} t \right) \right|_{T/8}^{T} \]

\[ = \frac{8}{T} \left. \frac{10 \sin \left( n_{o} t \right)}{2} \right|_{0}^{T/8} \]

\[ = \frac{40}{n} \sin \left( \frac{2n}{T} \right) \sin \left( \frac{8}{T} \right) \sin \left( \frac{0}{T} \right) \]

\[ a_n = \frac{40}{n} \sin \left( \frac{2n}{4} \right) \quad n \text{ off} \]
2a. (Continued)

From plot:

\[ n = 0, 1, 2, 3 \]

\[ \sin \left( \frac{n\pi}{4} \right) = 0, \quad \frac{\sqrt{2}}{2}, \quad 1, \quad \frac{\sqrt{2}}{2} \]

\[ n = 4, 5, 6, 7, \ldots \]

Therefore, odd coefficients are:

\[ a_1 = \frac{\sqrt{2}}{2} \cdot \frac{40}{\sqrt{2}}, \quad a_3 = \frac{\sqrt{2}}{2} \cdot \frac{40}{3\sqrt{2}}, \quad a_5 = \frac{\sqrt{2}}{2} \cdot \frac{40}{5\sqrt{2}} \]

\[ a_v = \text{ave value of } v_1(t) = 0 \text{ since equal positive and negative areas are under the } v_1(t) \text{ curve.} \]

\[ v_1(t) = \sum_{n \text{ odd } > 0} 40 \sin \left( \frac{n\pi}{4} \right) \cos (n\omega_0 t) \]

Now we convert to phasor form.

\[ a_n \cos (n\omega_0 t) + b_n \sin (n\omega_0 t) \text{ is time-domain rectangular representation} \]

\[ \text{P[ ] convert to rectangular phasor form} \]

\[ a_n \text{ (or } a_n \square 0^\circ) + jb_n \text{ (or } b_n \square 90^\circ) \]

Recall that phasor for \( \sin(\omega t) \) is \(-j\) or \( 1 \square 90^\circ \). Thus, our phasor is \( a_n - jb_n \).

Convert to polar form.

\[ A_n < \square \]

\[ A_n = \sqrt{a_n^2 + b_n^2} \]

\[ \square = \tan \left( \frac{b_n}{a_n} \right) \]

Thus, the polar form is:

\[ A_n = \sqrt{a_n^2 + b_n^2} < \frac{b_n}{a_n} \]
2a. (Continued)

Here, however, all $b_n = 0$. So we have $A_n = a_n$, $\phi_n = 0^\circ$. In other words, we have only $\cos(\ )$ terms, and the phase angle for $\cos(\ )$ terms in zero.

- $A_1 = a_1 = \frac{20\sqrt{2}}{\Box}$, $\phi_1 = 0^\circ$ $A_2 = a_2 = 0$, $\phi_2 = 0^\circ$
- $A_3 = a_3 = \frac{20\sqrt{2}}{3\Box}$, $\phi_3 = 0^\circ$ $A_4 = a_4 = 0$, $\phi_4 = 0^\circ$
- $A_5 = a_5 = \frac{20\sqrt{2}}{5\Box}$, $\phi_5 = 0^\circ$

**Note:** You may find it easier to derive symmetry results by drawing $v_i(t)$ and the $\cos(\ )$ or $\sin(\ )$ waveforms on a plot and multiplying them point by point (a rough sketch will do). The area under the curve corresponds to

$$\frac{T}{0} v_i(t) \cos(\ ) \quad \text{or} \quad \frac{T}{0} v_i(t) \sin(\ ) .$$

If the positive and negative areas cancel, $a_n$ (or $b_n$) = 0.

b. We want a band reject filter with center frequency $= \omega_o = 1\text{M rad/s}$, (see diagram in problem statement), and bandwidth $= 10\text{k rad/s}$ (see problem statement).

Our transfer function is $\frac{V_o(s)}{V_i(s)}$.

We use $V$-divider formula for $V_o(s)$ in terms of $V_i(s)$.

$$V_o(s) = V_i(s) \cdot \frac{Z_L}{1k\Box + Z_L} \quad Z_L = \text{components in box}$$
2b. (Continued)

\[ \frac{V_o(s)}{V_i(s)} = \frac{Z_L}{1k\Omega + Z_L} \]

To get \( \frac{V_o(s)}{V_i(s)} = \frac{j\omega = j1M\Omega/s}{j\omega = j1M\Omega/s} = 0 \), we need \( Z_L = 0 \) at \( \omega = 1M \).

We use an L in series with a C to get \( z \) cancellation:

\[ Z_L = j\omega L - \frac{j}{\omega C} \]

To get cancellation, \( L = + \frac{1}{C} \) at \( \omega = 1M \) or

\[ LC = \frac{1}{\omega^2} = \frac{1}{(1M)^2} = 1 \text{ ps} \]

We have RLC in series, and Text p. 734 tells us that for a series RLC band reject filter, we have \( \frac{R}{L} = \frac{R}{L} = 0.1H \).

Knowing \( L \), we can now solve for \( C \):

\[ C = \frac{1}{L\omega^2} = \frac{1}{0.1H(1M/s)^2} \]

\[ C = 10 \text{ pF} \]
The initial energy stored in the circuit is zero.

a. Choose values of $R_1$ and $C$ to accomplish the following:

1. $v(t)$ and $i(t)$ are decaying sinusoids $90^\circ$ out of phase with each other.

2. $\frac{1}{\tau} = T$, where $\tau$ is the exponential decay constant and $T$ is the period of oscillation of the decaying sinusoid.

b. Given $R_1$ and $C$ from (a), write numerical expressions for $v(t)$ and $i(t)$.

Answer:

a. $R_1 = 500 \text{ } \Omega$, $C = 32.4 \text{ } \mu \text{F}$

b. $v(t) = 159 e^{-61.7t} \sin (388t) \text{ mV}$

$i(t) = 2 e^{-61.7t} [\cos (388t) \text{ mA}]$

Solution:

a. Laplace transform circuit. Since initial energy is stored in the circuit, we have no extra current or voltage sources in the component models. In general, we would have to include such sources to account for the initial conditions.
3a. (Continued)

s-domain model:

\[ \frac{V_s}{s} = \frac{1}{s} V \]

First, we find \( V \).

Our calculations are cleaner if we use a Thevenin equivalent for the \( V \) source, \( R_1 \), and \( R_2 \).

\[ R_2 + \frac{1}{sC} \]

\[ \frac{1}{sC} \]

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3a. (Continued)

\[ \frac{1}{sC} sL = \frac{sL}{sC} + \frac{1}{sC} sL = \frac{sL}{1 + s^2 LC} = \frac{s}{s^2 + \frac{1}{LC}} \]

Use V-divider to find V:

\[ V = \frac{1}{sC} \frac{sL}{sC + R_1 \| R_2} \cdot V_g \frac{R_1}{R_1 + R_2} \]

Now simplify the expression.

\[ V = V_g \frac{R_1}{R_1 + R_2} \frac{sL}{1 + s^2 LC + R_1 \| R_2} = V_g \frac{R_1}{R_1 + R_2} \frac{SL}{SL + R_1 \| R_2 \left( 1 + s^2 LC \right)} \]
\[ = V_g \frac{R_1}{R_1 + R_2} \frac{sL}{s^2 \left( R_1 \| R_2 \right) LC + Ls + R_1 \| R_2} \]
\[ = V_g \frac{R_1}{R_1 + R_2} \frac{s}{s^2 + \frac{1}{\left( R_1 \| R_2 \right) C} s + \frac{1}{LC}} \]

Define \( R[] = R_1 \| R_2 \) and substitute \( V_g = \frac{1}{s} \):

\[ V = \frac{R_1}{R_1 + R_2} \frac{1}{R[] C} \frac{1}{s^2 + \frac{1}{R[] C} s + \frac{1}{LC}} \]

Observe that

\[ \frac{R_1}{R_1 + R_2} = \frac{R_1 \| R_2}{R_2} = \frac{R[]}{R_2} \]
\[ V = \frac{1}{R_2 C} \frac{1}{s^2 + \frac{1}{R[] C} s + \frac{1}{LC}} \]

\( V \) is of form \( \frac{k[]}{(s + [])^2 + []^2} = \mathcal{L}\{ke^{at} \sin (bt)\} \) where \( k = \text{real constant} \).
3a. (Continued)

I being 90° out of phase with V means i(t) must be of form ± k_2 e^{-at} \cos (ωt)

since sin (ωt ± 90°) = ± cos (ωt). Therefore, we want

\[ I = \mathcal{L}(k_2 e^{-at} \cos (ωt)) = \frac{k_2 (s + ω)}{(s + ω)^2 + ω^2} \]

where k_2 = real constant (perhaps < 0).

Before going further, we identify ω and ω from the expression for V:

\[ V = \frac{1}{R_2 C} s^2 + \frac{1}{R} s + \frac{1}{LC} = \frac{k}{(s + ω)^2 + ω^2} = \frac{k}{s + 2ωs + ω^2 + ω^2} \]

By matching the denominator polynomial coefficients, we get:

\[ 2ωs = \frac{1}{R} s \quad \omega = \frac{1}{2R} \]

\[ ω^2 + ω^2 = \frac{1}{LC} s \quad \omega = \sqrt{\frac{1}{LC} ω^2} \]

We also have k_2 = \frac{1}{R_2 C} as scaling factor. Therefore, k = \frac{1}{R_2 C}.

Now we find I:

\[ I = \frac{V}{R_1} \frac{1}{s C} \]

\[ R_1 \frac{1}{s C} = \frac{R_1}{s C} + \frac{1}{s C} = \frac{R_1}{1 + sR_1C} = \frac{1}{s + \frac{1}{R_1C}} \]

Therefore,

\[ I = V \frac{s + \frac{1}{R_1C}}{\frac{1}{C}} = VC \frac{s + \frac{1}{R_1C}}{s + \frac{1}{R_1C}} \]

\[ = \frac{1}{R_2} \frac{s + \frac{1}{R_1C}}{s^2 + \frac{1}{R} s + \frac{1}{LC}} \]
3a. (Continued)

Above, we concluded that \( I \) must be of form

\[
I = \frac{k_2(s + \Box)}{(s + \Box)^2 + \Box^2}
\]

The denominator polynomial in \( s \) in correct form. Matching the numerators gives:

\[
\frac{1}{R_2} s + \frac{1}{R_1C} = k_2(s + \Box)
\]

We must have \( k_2 = \frac{1}{R_2} \) and \( \frac{1}{R_1C} = \Box \). Earlier, we found

\[
\Box = \frac{1}{2R_1C} = \frac{1}{2R_1R_2C}
\]

We conclude that \( R_1C = 2(R_1R_2)C \), or \( R_1 = 2(R_1R_2) \).

\[
R_1 = R_2 \text{ gives } R_1R_2 = (R_1R_1) = \frac{1}{2}R_1 \text{ or } R_1 = 2(R_1R_2).
\]

\[
R_1 = R_2 = 500\square \quad \square = \frac{1}{R_1C} = \frac{1}{500\square \cdot C}
\]

We also must satisfy \( \frac{1}{\Box} = T \text{ (see problem statement). Now } \Box = 2\Box \).

From earlier,

\[
\Box^2 + \Box^2 = \frac{1}{LC} \quad \Box^2 + (2\Box)^2 = \frac{1}{LC}
\]

\[
\frac{1}{R_1C} \left[ 1 + (2\Box)^2 \right] = \frac{1}{LC} \quad \text{from} \quad \Box = \frac{1}{R_1C} \quad \text{or} \quad \frac{L[1 + (2\Box)^2]}{R_1^2} = C
\]
\[ C = \frac{0.2 \left(1 + 4 \pi^2 \right)}{(0.5 \, k)^2} = \frac{0.2 \left(1 + 4 \pi^2 \right)}{0.25} \, \text{mF} \]

\[ C = 0.8 \, \text{(40.5)} \, \text{mF} = 32.4 \, \text{mF} \]

A standard value for R would be 510 Ω.
A standard value for C would be 33 \text{ mF}.

From (a) we know

\[ v(t) = k \, e^{-\alpha t} \sin (\omega t) \quad \text{V} \leftarrow \text{part of } k \]

\[ i(t) = k_2 \, e^{-\alpha t} \cos (\omega t) \quad A \leftarrow \text{part of } k_2 \]

\[ \square = \frac{1}{R_1 C} \quad \square = 2 \square \]

\[ k = \frac{1}{R_2 C} = \frac{1}{R_1 C} \quad \text{since} \quad R_1 = R_2 \quad \text{or} \quad k = \frac{\square}{2 \square} = \frac{1}{2} \cdot \text{square} = \frac{1}{2} \]

\[ k_2 = \frac{1}{R_2} = \frac{1 \, \text{V}}{500 \, \text{mF}} \]

Numerical values:

\[ \square = \frac{1}{500 \, \text{mF} \cdot 32.4 \, \text{mF}} = 61.7/\text{s} \]

\[ \square = 2 \square (62.7)/\text{s} = 388 \, \text{rad/s} \]

\[ k = \frac{1}{2 \square} = 159 \, \text{mV} \]

\[ k_2 = \frac{1 \, \text{V}}{500 \, \text{mF}} = 2 \, \text{mA} \]

\[ v(t) = 159 \, e^{-61.7t} \sin (388t) \, \text{mV} \]

\[ i(t) = 2 \, e^{-61.7t} \cos (388t) \, \text{mA} \]
3a. (Continued)

Check:

\[
i(t) = i_{R_1} + i_C = \frac{v(t)}{R_1} + C \frac{dv(t)}{dt} = \frac{k}{R_1} e^{-\alpha t} \sin(\omega t) + C \frac{dk}{dt} e^{-\omega t} \sin(\omega t)
\]

\[
= \frac{k}{R_1} e^{-\alpha t} \sin(\omega t) + C k \left[ (-\alpha) e^{-\alpha t} \sin(\omega t) + e^{-\omega t} \cos(\omega t) \right]
\]

\[
= k \left[ \frac{1}{R_1} + \frac{k}{\omega} \left( -\alpha = -\frac{1}{R_1} \right) \right] e^{-\alpha t} \sin(\omega t) + k \left( k = \frac{1}{\omega} \right) e^{-\omega t} \cos(\omega t)
\]

\[
= \frac{1}{R_1} e^{-\alpha t} \cos(\omega t) = i(t) \quad (\text{since } R_1 = R_2)
\]
Find the z parameters for the 2-port network.

Answer:

\[ z_{11} = 5 \, \Omega \], \quad z_{12} = 10 \, \Omega \], \quad z_{21} = 10 \, \Omega \], \quad z_{22} = 20 \, \Omega \]

Solution:

Transformer is ideal, but \( I_2 \) in diagram is flowing into dot on secondary. The current flowing out of the dot is \( I_b \) in our equations for the transformer, and the current flowing into the dot on the primary side is \( I_a \) in our equations for the transformer:

\[
\frac{V_1}{V_2} = \frac{N_1}{N_2} \frac{I_b}{I_a} = \frac{N_2}{N_1}
\]

Note: Because of currents flowing in \( z_1 \) and \( z_2 \), \( I_b \neq I_1 \) and \( I_a \neq I_2 \).

\[
z_{11} = \frac{V_1}{I_1} \bigg|_{I_2=0} = \text{impedance seen looking into input side with no connection to output side, (although } z_c \text{ is still in circuit)}
\]

We use formula for reflected impedance of secondary into primary:

\[
z_r = \frac{N_1^2}{N_2^2} z_2 \quad (\text{since } z_2 \text{ acts like } z_L)
\]
4. (Continued)

Our model, given $\frac{N_1}{N_2} = \frac{1}{2}$ turns ratio is:

$$z_{11} = z_1 \left| \begin{array}{c} z_r = (5 - j5) \| (5 + j5) \\ z_2 = (5 - j5)(5 + j5) \\ z_3 = 5^2 + 5^2 = 50 \end{array} \right|$$

For $z_{22}$ we use the same approach, but we reverse the roles of primary and secondary.

$$z_{22} = \frac{V_2}{I_2} \left| \begin{array}{c} z_1 = 0 \\ z_r = \left( \begin{array}{c} N_2 \\ N_1 \end{array} \right) z_1 \end{array} \right|$$

Circuit model:

$$z_{22} = \text{impedance seen looking into output terminals}$$

$$= z_r \left| \begin{array}{c} z_2 = (20 - j20) \| (20 + j20) = 20 \cdot (1 - j) \| (1 + j) \end{array} \right|$$

$$z_{22} = 20 \left( \frac{(1 - j)(1 + j)}{1 - j + 1 + j} = 20 \frac{2}{2} = 20 \right|$$
4. (Continued)

For \( z_{12} \), we have

\[
\frac{V_1}{I_2} = \frac{V_1}{I_1} = 0
\]

We use the same model as for \( z_{22} \) to get an equation relating \( V_2 \) and \( I_2 \):

\[
V_2 = I_2 \cdot z_{r2} = I_2 z_{22} = I_2 \cdot 20\Omega
\]

From ideal transformer equations, \( \frac{V_1}{V_2} = \frac{N_1}{N_2} \).

\[
\frac{V_1}{N_2} = I_2 \cdot z_{22} = \frac{N_1}{N_2} = I_2 20\Omega \frac{1}{2} = I_2 10\Omega
\]

or

\[
z_{12} = \frac{V_1}{I_2} = 10\Omega
\]

For \( z_{21} = \frac{V_2}{I_1} \), we use the same model as for \( z_{11} \). We have

\[
V_1 = I_1 z_{r1} = I_1 z_{11} = I_1 5\Omega
\]

\[
\frac{V_1}{V_2} = \frac{N_1}{N_2} \quad \text{or} \quad V_2 = \frac{N_2}{N_1} = I_1 z_{11} = \frac{N_2}{N_1} = I_1 5\Omega \frac{2}{1}
\]

\[
z_{21} = \frac{V_2}{I_1} = 5\Omega \cdot \frac{2}{1} = 10\Omega
\]

**Comment:** Because \( V_1 \) and \( V_2 \) were defined with + signs at transformer dots, we were able to avoid problems with signs of currents. We only needed the ideal transformer equations for voltages and reflected impedances.