State Variables and MATLAB ode45

Time-domain analysis of circuits with more than one L and C is difficult because it requires solution of characteristic equations higher than second degree. This supplement illustrates the use of MATLAB functions, ode23 and ode45, for solving a system of coupled first-order differential equations of the form

\[ \frac{dx}{dt} = f(x,t) \]

where \( x \) is a vector called the state vector, and \( t \) is the independent variable, which in our case will be time. As indicated in Mastering MATLAB 6, Chapter 24, a differential equation can always be expressed as a system of coupled first-order differential equations. The MATLAB functions are powerful because they can be used to solve nonlinear as well as linear differential equations.

Consider the third-order circuit in Fig. 1 as an example for illustrating the use of the state vector (state variables) and ode45. Let's use MATLAB to plot \( v_2 \) as a function of time. Using Kirchhoff's laws in the time domain, we could write the third-order differential equation for \( v_2 \) and then convert it to three coupled first-order equations following the procedure outlined in the MATLAB book. It is much easier, however, to obtain the system of coupled first-order equations directly in terms of what are called the state variables. For circuits, the state variables are the currents through inductances and the voltages across capacitances. Using Kirchhoff's laws in the time domain, it is easy to write first-order differential equations in terms of these variables because \( i = C \frac{dv}{dt} \) for a capacitance, and \( v = L \frac{di}{dt} \) for an inductance, both of which involve first derivatives.

* The material in this handout is based extensively on concepts developed by C. H. Durney, Professor Emeritus of the University of Utah.
Fig. 1. Third-order circuit. \( v_g(t) \) is a step function that switches from \(-v_0\) to \(v_0\) at \(t = 0\).

The state variables for the circuit in Fig. 1 are \(v_1, v_2,\) and \(i_1\). The first equation is easily obtained from the relationship between the voltage and current for \(C_1\):

\[
dv_1/dt = i_1/C
\]  

The second equation is obtained by writing Kirchhoff’s voltage equation around the closed path \(abcdea\):

\[-v_g + v_1 + L di_1/dt + v_2 = 0\]

from which we get

\[
di_1/dt = -v_1/L - v_2/L + v_g/L
\]  

The third equation is obtained by using Kirchhoff’s current law,

\[i_1 = i_2 + i_3\]

and \(i_3 = v_2/R, i_2 = C_2 dv_2/dt\) to get

\[
dv_2/dt = i_1/C_2 - v_2/RC_2
\]  

Now defining the state vector (note that \(x\) is a column vector) as

\[
x = [v_1, i_1, v_2]
\]  

we get the three coupled first-order differential equations from (4), (1), (2), and (3):
\[
\frac{dx_1}{dt} = \frac{x_2}{C_1} \tag{5}
\]
\[
\frac{dx_2}{dt} = -\frac{x_1}{L} - \frac{x_3}{L} + \frac{v_g}{L} \tag{6}
\]
\[
\frac{dx_3}{dt} = \frac{x_2}{C_2} - \frac{x_3}{RC_2} \tag{7}
\]

Before writing the Matlab® program, we need the initial conditions on the vector x. The initial conditions for x are much easier to obtain than for the third-order differential equation for v2. Because the current cannot change instantaneously through an inductance, and because the voltage across a capacitance cannot change instantaneously,

\[
v_1(0+) = v_1(0-) = -v_0
\]
\[
i_1(0+) = i_1(0-) = 0
\]
\[
v_2(0+) = v_2(0-) = 0
\]

The initial value of x is thus

\[
x0 = [-v_0 \ 0 \ 0]'
\tag{8}
\]

The ode45 function requires a function that calculates xdot (which stands for dx/dt) as a function of t and x (see p. 263 in the MATLAB® book). Here is a function that calculates xdot as described by (5)-(7):

```matlab
function xdot = ckt(t,x)
    vg=1;
    C1=100e-9;
    C2=100e-9;
    R=30;
    L=184e-6;
    xdot(1) = x(2)/C1;
    xdot(2) = -(x(1) + x(3))/L + vg/L;
    xdot(3) = x(2)/C2 - x(3)/(R*C2);
    xdot = [xdot(1); xdot(2); xdot(3)];
```

To solve for x and plot v2, we can now type the following in the command window:

```matlab
t0=0; tf=60e-6;
```
tspan=[t0,tf];
x0=[-1 0 0]';
[t,x]=ode45('ckt',tspan,x0);
plot(t,x(:,3))

We could also plot all three components of x by using plot(t,x), as shown in Fig. 2.

If tf is chosen to be too large, the matrix size limitation in the student edition of MATLAB® will be exceeded. Also, sometimes choosing tf to be too large will cause a message "singularity expected," and the function will not be executed.

Fig.2. Graph of the state vector x versus time for the circuit of Fig. 1.
Another practice problem:

Use `ode45` to solve for x for the circuit in Fig. 1.1 of the Unit 1 Study Guide when

\[ R_1 = 20 \, \Omega, \quad R_2 = 5 \, \Omega, \quad C_1 = 100 \, \text{nF}, \quad C_2 = 200 \, \text{nF}, \quad L = 300 \, \mu\text{H}, \quad v_0 = 1 \, \text{V}. \]

Then plot \( i_1 \) versus \( t \) and \( i_3 \) versus \( t \).

Answer: