Class 7: Linear, Time-Invariant Systems

1. Linear, Time-Invariant Systems
   1. Continuous-time convolution
   2. Discrete-time convolution
Continuous-time Convolution
Convolution (Analytical)

- $x(t) = u(t + 2) - u(t - 2)$
- $h(t) = u(t + 2) - 2u(t) + u(t - 2)$
- Compute $x(t) \ast h(t)$

\[
\begin{align*}
\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-\infty}^{\infty} [u(\tau + 2) - u(\tau - 2)][u((t - \tau) + 2) - 2u(t - \tau) + u((t - \tau) - 2)]d\tau \\
&= \int_{-\infty}^{\infty} u(\tau + 2)u((t - \tau) + 2) - u(\tau + 2)2u(t - \tau) + u(\tau + 2)u((t - \tau) - 2)d\tau \\
&\quad + \int_{-\infty}^{\infty} -u(\tau - 2)u((t - \tau) + 2) + u(\tau - 2)2u(t - \tau) - u(\tau - 2)u((t - \tau) - 2)d\tau \\
&= \int_{-2}^{t+2} u(t + 4) \, d\tau - \int_{-2}^{t} 2u(t + 2) \, d\tau + \int_{-2}^{t-2} u(t) \, d\tau \\
&\quad - \int_{2}^{t+2} u(t) \, d\tau + \int_{2}^{t} 2u(t - 2) \, d\tau - \int_{2}^{t-2} u(t - 4) \, d\tau \\
&= (t + 4)u(t + 4) - 2(t + 2)u(t + 2) + 2(t - 2)u(t - 2) - (t - 4)u(t - 4)
\end{align*}
\]
Convolution (Graphical)

\[ x(t) = x(\tau) \]

\[ h(t) \]

\[ h(-\tau + 0) \]

\[ y(t) \]
Convolution (Graphical)

\[ x(t) = x(\tau) \]

\[ y(t) = x(t) \ast h(t) \]

\[ h(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 2 \\ -1 & \text{if } -2 \leq t < 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ h(-\tau + 0) \]

\[ t = 0 \]

\[ t = -10 \]
Convolution (Graphical)

\[ x(t) = x(\tau) \]

\[ h(t) \]

\[ h(-\tau + 0) \]

\[ y(t) \]
Convolution (Graphical)

\[ x(t) = x(\tau) \]

\[ h(t) \]

\[ h(-\tau + 0) \]

\[ t = 0 \]

\[ t = -4 \]

\[ h(-\tau - 4) \]

\[ y(t) \]
Convolution (Graphical)

\[ x(t) = x(\tau) \]

\[ h(t) \]

\[ h(-\tau + 0) \]

\[ y(t) \]

Multiply & Integrate = 1

\( t = 0 \)

\( t = -3 \)
Convolution (Graphical)

\[ x(t) = x(\tau) \]

\[ h(t) \]

\[ h(-\tau + 0) \]

\[ t = 0 \]

\[ t = -2 \]

\[ h(-\tau - 2) \]

Multiply & Integrate = 2

\[ y(t) \]
Convolution (Graphical)

\[ x(t) = x(\tau) \]

\[ h(t) \]

\[ h(-\tau + 0) \]

\[ t = 0 \]

\[ t = -1 \]

\[ h(-\tau - 1) \]

Multiply & Integrate = 1

\[ y(t) \]
Convolution (Graphical)

\[ x(t) = x(\tau) \]

\[ h(t) \]

\[ h(-\tau + 0) \]

\[ y(t) \]

Multiply & Integrate = 0

\[ t = 0 \]
Convolution (Graphical)

\[ x(t) = x(\tau) \]

\[ h(t) \]

\[ h(-\tau + 0) \]

Multiply & Integrate = -1

\[ y(t) \]
Convolution (Graphical)

\[ x(t) = x(\tau) \]

\[ h(t) \]

\[ h(-\tau + 0) \quad t = 0 \]

\[ t = 2 \]

\[ h(-\tau + 2) \]

\[ y(t) \]

Multiply & Integrate = -2
Convolution (Graphical)

\[ x(t) = x(\tau) \]

\[ h(t) \]

\[ h(-\tau + 0) \]

\[ y(t) \]

Multiply & Integrate = -1
Convolution (Graphical)

\[ x(t) = x(\tau) \]

\[ y(t) = \int_{-2}^{2} x(\tau) h(t-\tau) \, d\tau \]

\[ h(t) \]

\[ h(-\tau + 0) \]

\[ t = 0 \]

\[ t = 4 \]
Convolution (Graphical)

\[ x(t) = x(\tau) \]

\[ h(t) \]

\[ h(-\tau + 0) \]

\[ y(t) \]
Discrete-time Convolution
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ y[n] \]
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ y[n] \]
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ y[n] \]
Convolution (Graphical)

$x[n] = x[m]$

$h[n]$  

$h[-m + 0]$  

$y[n]$  

$n = 0$

$n = -10$
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ y[n] \]

\[ n = 0 \]

\[ n = -9 \]
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ n = 0 \]

\[ n = -8 \]

\[ h[-m - 8] \]

\[ y[n] \]
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ n = 0 \]

\[ n = -7 \]

\[ h[-m - 7] \]

\[ y[n] \]
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ n = 0 \]

\[ n = -6 \]

\[ h[-m - 6] \]

\[ y[n] \]
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ n = 0 \]

\[ n = -5 \]

\[ h[-m - 5] \]

\[ y[n] \]
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ n = 0 \]

\[ n = -4 \]

Multiply & Sum = 1

\[ y[n] \]
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ n = 0 \]

\[ n = -3 \]

Multiply & Sum = 2

\[ y[n] \]
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ n = 0 \]

\[ n = -2 \]

Multiply & Sum = 2

\[ y[n] \]

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Convolution (Graphical)

$x[n] = x[m]

h[n]

h[-m + 0]

$y[n] = \sum h[n] x[m-n]$

Multiply & Sum = 1

$n = 0$

$n = -1$

Multiply & Sum = 1
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ \sum \text{Multiply & Sum} = 0 \]
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ n = 0 \]

\[ n = 1 \]

Multiply & Sum = -1

\[ y[n] \]
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ y[n] \]

Multiply & Sum = -2
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

Multiply & Sum = -2

\[ n = 0 \]

\[ n = 3 \]
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ n = 0 \]

\[ n = 4 \]

Multiply & Sum = -1

\[ y[n] \]
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ n = 0 \]

\[ n = 5 \]

\[ h[-m + 5] \]
Convolution (Graphical)

Let $x[n] = x[m]$ and $h[n]$ be the input and impulse response, respectively.

The convolution $y[n]$ is calculated as:

$$y[n] = x[n] * h[n]$$

For $n = 0$,

$$y[0] = x[0] * h[0] = 1 * 1 = 1$$

For $n = 6$,


Thus, the convolution $y[n]$ is shown graphically with the impulses corresponding to $x[n]$ and $h[n]$ superimposed.
Convolution (Graphical)

\[ x[n] = x[m] \]

\[ h[n] \]

\[ h[-m + 0] \]

\[ n = 0 \]

\[ n = 7 \]

\[ h[-m + 7] \]