1. A continuous random variable $X$ has the probability density function:

$$f(x) = \begin{cases} 
0, & x < 1 \\
hx - h, & 1 \leq x \leq 2 \\
3h - hx, & 2 \leq x \leq 3 \\
0, & x > 3
\end{cases}$$

which can be graphed as

(a) Find $h$ which makes $f(x)$ a valid probability density function.

The area underneath the triangle is $\frac{1}{2}(3 - 1)h = h$ which must equal 1 to satisfy $\int_{-\infty}^{\infty} f(x)dx = 1$. Therefore $h = 1$. You can of course solve this problem by explicitly integrating the function $f(x)$ but it is much simpler to use geometry and the area of the triangle.

(b) Find the cumulative distribution function $F(x)$.

First write down the density function after substituting $h = 1$:

$$f(x) = \begin{cases} 
0, & x < 1 \\
x - 1, & 1 \leq x \leq 2 \\
3 - x, & 2 \leq x \leq 3 \\
0, & x > 3
\end{cases}$$

There are 4 cases:

- $x < 1$: Here $F(x) = 0$.
- $1 \leq x \leq 2$:

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{1} 0dt + \int_{1}^{x} (t - 1)dt$$
\[
2 \leq x \leq 3:
\]
\[
F(x) = \int_{-\infty}^{x} f(t) \, dt
\]
\[
= \int_{-\infty}^{1} 0 \, dt + \int_{1}^{2} (t - 1) \, dt + \int_{2}^{x} (3 - t) \, dt
\]
\[
= F(2) + \int_{2}^{x} (3 - t) \, dt
\]
\[
= \left( \frac{2^2}{2} - 2 + \frac{1}{2} \right) + 3t - \frac{t^2}{2} \bigg|_{x}^{2}
\]
\[
= \frac{1}{2} + 3x - 6 - \frac{x^2}{2} + 2 = -\frac{x^2}{2} + 3x - \frac{7}{2}
\]

Notice that \( F(3) = 1 \) which is what is required.

\[
\bullet \quad x > 3: \quad \text{Here} \quad F(x) = 1
\]

So

\[
F(x) = \begin{cases} 
0, & x < 1 \\
\frac{x^2}{2} - x + \frac{1}{2}, & 1 \leq x \leq 2 \\
-\frac{x^2}{2} + 3x - \frac{7}{2}, & 2 \leq x \leq 3 \\
1, & x > 3
\end{cases}
\]

2. Random variable \( X \) has a normal probability distribution with mean 10.3 and standard deviation 2.

(a) Compute the numerical value of \( P(7.2 \leq X \leq 13.8) \).

We first convert to a standard normal distribution with \( Z = \frac{X - 10.3}{2} \). When \( X = 7.2 \), \( Z = \frac{7.2 - 10.3}{2} = -1.55 \) and \( X = 13.8 \), \( Z = \frac{13.8 - 10.3}{2} = 1.75 \). Therefore,

\[
P(7.2 \leq X \leq 13.8) = P(-1.55 \leq Z \leq 1.75) = P(Z \leq 1.75) - P(Z \leq -1.55)
\]

From the attached table we find that \( P(Z \leq 1.75) = 0.9599 \) and \( P(Z \leq -1.55) = 0.0606 \). So the answer is \( 0.9599 - 0.0606 = 0.8993 \).

(b) Find a value \( d \) such that \( X \) is in the range \( 10.3 \pm d \) with probability 0.999.

We want \( P(10.3 - d \leq X \leq 10.3 + d) = 0.999 \). Again converting to standard normal distribution: when \( X = 10.3 - d \), \( Z = \frac{10.3 - d - 10.3}{2} = -0.5d \) and when \( X = 10.3 + d \), \( Z = \frac{10.3 + d - 10.3}{2} = 0.5d \). So we are looking for a \( d \) such that \( P(-0.5d \leq Z \leq 0.5d) = 0.999 \):

\[
P(-0.5d \leq Z \leq 0.5d) = 1 - (P(Z < -0.5d) + P(Z > 0.5d))
\]
\[
0.999 = 1 - 2P(Z < -0.5d)
\]
\[
P(Z < -0.5d) = \frac{1 - 0.999}{2} = 0.0005
\]
From the attached table we find that \(-0.5d = -3.3\), so \(d = 6.6\).

(c) **Let \(Y\) be a random variable with variance \(\sigma_Y^2 = 6\) and independent of \(X\). Compute the variance of \(5X - 3Y\).**

\(5X - 3Y\) is a linear combination of the random variables \(X\) and \(Y\). The variance of \(X\) is 4 (the square of its standard deviation). Using the fact that \(X\) and \(Y\) are independent we find that

\[
\sigma_{5X-3Y}^2 = 25\sigma_X^2 + 9\sigma_Y^2 = 25 \times 4 + 9 \times 6 = 154
\]

3. Let \(X\) and \(Y\) be two continuous random variables with the joint density function

\[
f(x, y) = \begin{cases} x + y, & 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0, & \text{elsewhere} \end{cases}
\]

(a) **Are the random variables \(X\) and \(Y\) independent? Justify your answer.**

Let’s compute the marginal density functions:

\[
g(x) = \int_{y=-\infty}^{\infty} f(x, y) \, dy = \int_{y=0}^{y=1} (x + y) \, dy = xy|_{y=1}^{y=0} + \int_{y=0}^{y=1} \frac{y^2}{2} \, dy = x + \frac{1}{2}
\]

\[
h(y) = \int_{x=-\infty}^{\infty} f(x, y) \, dx = \int_{x=0}^{x=1} (x + y) \, dx = \frac{x^2}{2} \bigg|_{x=0}^{x=1} + xy|_{x=0}^{x=1} = \frac{1}{2} + y
\]

For independence we must have \(f(x, y) = g(x)h(y)\), which doesn’t hold in this case. So \(X\) and \(Y\) are NOT independent.

(b) **Compute the numerical value of \(P(Y \geq \frac{1}{2}, X \leq \frac{1}{2})\).**

\[
P(Y \geq \frac{1}{2}, X \leq \frac{1}{2}) = \int_{y=1/2}^{y=1} \int_{x=0}^{x=1/2} (x + y) \, dx \, dy = \int_{y=1/2}^{y=1} \left( \frac{x^2}{2} \bigg|_{x=0}^{x=1/2} + xy|_{x=0}^{x=1/2} \right) \, dy
\]
4. Let $X$ be the sent bit and $Y$ the received bit in a binary communications channel. The joint probability distribution $f(x, y)$ is given as:

<table>
<thead>
<tr>
<th>$f(x, y)$</th>
<th>$x=0$</th>
<th>$x=1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y=0$</td>
<td>0.4</td>
<td>0.1</td>
</tr>
<tr>
<td>$y=1$</td>
<td>0.1</td>
<td>0.4</td>
</tr>
</tbody>
</table>

(a) **Compute the numerical value of** $P(Y = 1 | X = 0)$

$$P(Y = 1 | X = 0) = \frac{f(0, 1)}{g(0)} = \frac{0.1}{0.4 + 0.1} = 0.2$$

(b) **Compute the covariance of random variables** $X, Y$.

$$\sigma^2_{XY} = E[XY] - \mu_X \mu_Y;$$

therefore we first need to compute $\mu_X, \mu_Y$ and $E[XY]$.

$$\mu_X = 0 \times 0.4 + 0 \times 0.1 + 1 \times 0.1 + 1 \times 0.4 = 0.5$$

$$\mu_Y = 0 \times 0.4 + 1 \times 0.1 + 0 \times 0.1 + 1 \times 0.4 = 0.5$$

$$E[XY] = 0 \times 0.4 + 0 \times 0.1 + 0 \times 0.1 + 1 \times 0.4 = 0.4$$

Therefore

$$\sigma^2_{XY} = 0.4 - 0.5 \times 0.5 = 0.15$$

(c) **When a single bit is sent and received, we say that an error has occurred if $Y \neq X$. If a 8-bit long message is sent over this communication channel, what is the probability that 1 or less errors will occur?**

First find the probability of making an error when a single bit is sent:

$$P(X \neq Y) = P(X = 0, Y = 1) + P(X = 1, Y = 0) = 0.2$$

Each bit sent is a Bernoulli trial with $P(error) = 0.2$. Then the number of errors when 8 bits are sent follow a Binomial distribution $b(x; p = 0.2, n = 8)$. The probability of one or less errors in 8 bits is found as

$$\sum_{x=0}^{x=1} b(x; p = 0.2, n = 8) = 8 C_0 \times 0.2^0 \times 0.8^8 + 8 C_1 \times 0.2^1 \times 0.8^7 \approx 0.5033$$