1. A cellphone service provider wants to analyze the signal strengths of its network. Let $X$ be the random variable that is the distance (in miles) that a customer will be to the nearest cellphone tower. Analyzing their database, the company finds that the probability density function for $X$ is

$$f(x) = \begin{cases} \frac{hx}{2}, & 0 \leq x \leq 6 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find $P(1 \leq X \leq 5)$. Give your answer as a numerical value.
(b) Find the mean and standard deviation of the random variable $X$.

\[ \text{Area} = \frac{36h}{2} = 1 \quad \text{h} = \frac{1}{18} \]

\[ P(1 \leq X \leq 5) = \int_1^5 \frac{x}{18} \, dx = \frac{x^2}{36} \bigg|_1^5 = \frac{25 - 1}{36} = \frac{24}{36} = \frac{2}{3} \]

\[ \mu = \int_{-\infty}^{\infty} x f(x) \, dx = \int_0^6 x \frac{x}{18} \, dx = \frac{x^3}{54} \bigg|_0^6 = \frac{6^3}{54} = 4 \]

\[ \sigma^2 = E[X^2] - \mu^2 \]

\[ E[X^2] = \int_0^6 x^2 \frac{x}{18} \, dx = \frac{x^4}{18 \cdot 4} \bigg|_0^6 = 18 \]

\[ \sigma^2 = 18 - 4^2 = 2 \]

\[ \sigma = \sqrt{2} \]
2. An element has two electrons. Each of the electrons can occupy one of the three orbits of this element. Let the random variables $X$ and $Y$ denote the orbit number occupied by these two electrons. The joint probability distribution is given as:

<table>
<thead>
<tr>
<th>$f(x,y)$</th>
<th>$x=1$</th>
<th>$x=2$</th>
<th>$x=3$</th>
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<tbody>
<tr>
<td>$y=1$</td>
<td>0</td>
<td>1/8</td>
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<td>$y=2$</td>
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<td>$y=3$</td>
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(a) Are these random variables independent?
(b) Find $P(X \leq 2|Y = 2)$.

\[ a) \quad g(x) = \begin{cases} 
\frac{1}{8}, & x = 1 \\
\frac{3}{4}, & x = 2 \\
\frac{1}{8}, & x = 3 
\end{cases} \quad \text{and} \quad h(y) = \begin{cases} 
\frac{1}{8}, & y = 1 \\
\frac{3}{4}, & y = 2 \\
\frac{1}{8}, & y = 3 
\end{cases} \]

\[ g(x) \cdot h(y) = \frac{1}{8} \cdot \frac{1}{8} = \frac{1}{64} \]

\[ f(1,1) = 0 \]

\[ \text{NOT independent} \]

\[ b) \quad f_X(x|y) = \frac{f(x,y)}{h(y)} \]

\[ f_X(x|y=2) = \frac{f(x,y)}{h(2)} = \begin{cases} 
\frac{1}{6}, & x = 1 \\
\frac{2}{3}, & x = 2 \\
\frac{1}{6}, & x = 3 
\end{cases} \]

\[ P(X \leq 2|Y = 2) = \frac{1}{6} + \frac{2}{3} = \frac{5}{6} \]
3. Consider the circuit shown below:

![Circuit Diagram]

The currents $X$ and $Y$ are independent random variables and their marginal densities are given as:

- The current $X$ is a normal (Gaussian) random variable with mean 3 Amperes and standard deviation 0.5 Amperes.
- The current $Y$ has the marginal density function

$$h(y) = \begin{cases} 
0.5, & 0 \leq y \leq 2 \\
0, & \text{otherwise}
\end{cases} \quad \rightarrow \quad \mu_Y = 1$$

(a) Find the probability that the current $X$ is larger than 3.83 Amperes.
(b) The sum of two currents $X + Y$ flow over a into a 10 Ohm resistor. Therefore, the power dissipated over the resistor is $10(X+Y)^2$. Find the average power dissipated over the resistor.

\[
\begin{align*}
\text{a) } P(X > 3.83) &= P(Z > \frac{3.83 - 3}{0.5}) \\
&= P(Z > 1.66) = 1 - P(Z < 1.66) \\
&= 1 - 0.9515 = 0.0485 \\
\text{b) } E[10(X+Y)^2] &= E[10X^2 + 20XY + 10Y^2] \\
E[X^2] &= \sigma_X^2 + \mu_X^2 = 0.5^2 + 3^2 = 9.25 \\
\text{independent } E[XY] &= E[X]E[Y] = \mu_X \mu_Y = 3 \cdot 1 = 3 \\
E[Y^2] &= \int_0^2 y^2 \cdot 0.5 \, dy = \frac{y^3}{6} \bigg|_0^2 = 4/3
\end{align*}
\]

\[8. \text{b) Compute } 10 \times 9.25 + 20 \times 3 + 10 \times 4/3 = 165.83\]
4. A random number generator produces binary sequences that are 4-bits long. Each bit can be ON or OFF. Each bit is turned ON with a probability 0.5. The output of a summing circuit is equal to the number of bits that are ON in the 4-bit sequence. Let $X$ be the discrete random variable denoting the output of the summing circuit.

(a) What is the probability that the brightness level $X$ greater than or equal to 3?

(b) Another 4-bit random number generator drives another summing circuit whose output we will call $Y$. Again, each bit is ON with probability 0.5. $X$ and $Y$ are independent random variables. Find the probability that $Y = X + 3$.

\[ p(X \geq 3) = \sum_{x=3}^{4} b(x; n=4, p=0.5) \]

\[ = C_3^4 0.5^4 + C_4^4 0.5^4 \]

\[ = \frac{5}{16} \]

\[ b) \text{ Independence } f(x, y) = g(x) h(y) \]

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\[ p(Y = X + 3) = \text{This Area} = g(0)h(3) + g(1)h(4) \]

\[ = C_0 0.5^4 + C_3 0.5^4 \]

\[ = \frac{8}{256} \]