2 additional pages of
Notes on Chapter 4
For a two-port circuit, calculate (See also Ex. 4.5 Text p. 172):

a. the reflection coefficient $\Gamma_1$ at port 1 if port 2 is connected to a mismatched load of reflection $\Gamma_L$

b. the power delivered to this mismatched load in terms of input power $a_1 a_1^*$

**Solution**

**a.** The circuit diagram is given in Fig. 1:

\[
\begin{align*}
\begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix}
\begin{bmatrix}
a_1 \\
b_1
\end{bmatrix}
= \begin{bmatrix}
a_2 \\
b_2
\end{bmatrix}
\end{align*}
\]

For the mismatched load of reflection $\Gamma_L$, note that $a_2 = \frac{\Gamma_L}{b_2} \quad (3)$

Eq. (3) can be rewritten as $b_2 = \frac{a_2}{\Gamma_L} \quad (4)$

Substituting Eq. (4) into Eq. (2) and rearranging terms,

\[
S_{21} a_1 = \left( \frac{1}{\Gamma_L} - S_{22} \right) a_2 = \left( \frac{1}{\Gamma_L} - S_{22} \Gamma_L \right) a_2 \quad (5)
\]

Substitute Eq. (5) into Eq. (1), we can write

\[
\Gamma_1 = \frac{b_1}{a_1} = \left[ S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - S_{22} \Gamma_L} \right] \quad (6)
\]

b. Power delivered to the mismatched load $= b_2 b_2^* - a_2 a_2^* = b_2 b_2^* \left( 1 - \left| \frac{\Gamma_L}{\Gamma_L} \right|^2 \right) \quad (7)$

Expressing Eq. (2) in terms of $a_1$, we can write (by using Eq. (5))

\[
b_2 = \left[ S_{21} + \frac{S_{22} S_{21} \Gamma_L}{1 - S_{22} \Gamma_L} \right] a_1 = \left[ \frac{S_{21}}{1 - S_{22} \Gamma_L} \right] a_1 \quad (8)
\]

Combining Eqs. (7) and (8), we can write:

\[
\text{Power delivered to the mismatched load} = \left| \frac{S_{21}}{1 - S_{22} \Gamma_L} \right|^2 \left( 1 - \left| \frac{\Gamma_L}{\Gamma_L} \right|^2 \right) \quad (9)
\]

**Ex. 2**

Calculate the quantities $a_1$ and $b_1$ in Ex. 1 for a circuit for which the $S$-parameters are

\[
\begin{bmatrix}
0.1 & 0.8j \\
0.8j & 0.2
\end{bmatrix}
\]

and $\Gamma_L = 0.333$

**Solution**

a. From Eq. (6),

\[
\Gamma_1 = \frac{b_1}{a_1} = \left[ S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - S_{22} \Gamma_L} \right] = -0.1285
\]

Return loss $RL = 20 \log \left| \frac{b_1}{a_1} \right| = 17.82$ dB

b. From Eq. (9), power delivered to the load $= 0.654$

\[
= \frac{\text{input power } a_1 a_1^*}{\text{input power } a_1 a_1^*}
\]

i.e. 65.4% of the input power can be delivered to this mismatched load

c. For a perfectly matched load connected to the output port 2, $\Gamma_L = 0$ and power delivered to this perfectly matched load $= \left| S_{21} \right|^2 = (0.8)^2 = 0.64$
Similar to Prob. 4.17

A reciprocal lossless 4-port network

\[ a_4 = b_3 e^{-j100^\circ} \]
\[ a_3 = b_4 e^{-j100^\circ} \]

\[ [S] = \begin{bmatrix} 0.6 e^{j90^\circ} & 0 & 0 & 0.8 e^{-j90^\circ} \\ 0 & 0.7 e^{j45^\circ} & 0.7 e^{-j45^\circ} & 0 \\ 0 & 0.7 e^{-j45^\circ} & 0.7 e^{j45^\circ} & 0 \\ 0.8 e^{j90^\circ} & 0 & 0 & 0.6 e^{-j90^\circ} \end{bmatrix} \]

Note that parts 3 and 4 are connected with a lossless matched transmitting line of electrical length 100°

\[ \frac{b_2}{a_1} = \frac{0.8 e^{-j100^\circ} x 0.7 e^{j0^\circ}}{1 - \left(0.6 e^{j90^\circ} \right) \left(-e^{-j100^\circ}\right) \left(0.7 e^{j45^\circ}\right) \left(-e^{j100^\circ}\right)} \]

\[ = \frac{0.56 e^{-j145^\circ}}{1 - 0.42 e^{-j165^\circ}} \]

\[ \frac{b_2 b_2^*}{a_1 a_1^*} = \frac{(0.56)^2}{(1 - 0.42 e^{-j165^\circ})(1 - 0.42 e^{j165^\circ})} = \frac{(0.56)^2}{1 + (0.42)^2 - 2 \times 0.42 \cos 65^\circ} = 0.4648 \]

\[ \text{IL} = -10 \log(0.4648) = -3.33 \text{ dB} \]