Scattering Parameters

A set of parameters that are very useful in the microwave range are the scattering parameters. These parameters are defined in terms of traveling waves and completely characterize the behavior of a two-port Network.

Impedance, Admittance, Hybrid and ABCD matrices

At low frequency a 2-port Nlw can be represented as a impedance matrix, admittance matrix, hybrid matrix or a chain for ABCD matrix.

We can define these matrices as:

**Z parameters**

\[ V_1 = Z_{11} \cdot I_1 + Z_{12} \cdot I_2 \]
\[ V_2 = Z_{21} \cdot I_1 + Z_{22} \cdot I_2 \]
\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} =
\begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
\]

**Y parameters**

\[
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix} =
\begin{bmatrix}
y_{11} & y_{12} \\
y_{21} & y_{22}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
\]
h parameters

\[
\begin{bmatrix}
  v_1 \\
  i_2
\end{bmatrix} =
\begin{bmatrix}
  h_{11} & h_{12} \\
  h_{21} & h_{22}
\end{bmatrix}
\begin{bmatrix}
  i_1 \\
  v_2
\end{bmatrix}
\]

ABCD parameters

\[
\begin{bmatrix}
  v_1 \\
  i_1
\end{bmatrix} =
\begin{bmatrix}
  A & B \\
  C & D
\end{bmatrix}
\begin{bmatrix}
  v_2 \\
  -i_2
\end{bmatrix}
\]

These formulas are really useful at low frequencies because the parameters are readily measurable using short and open circuit tests at the terminal of the two port network. For example,

\[
Z_{11} = \left. \frac{v_1}{i_1} \right|_{i_2 = 0}
\]

is measured with an ac open circuit at port 2 (i.e. \(i_2 = 0\)).

Example

Consider a 2 port Network as shown in the figure below. Find the impedance parameters

\[
\begin{array}{c}
  I_1 \\
  V_1 \\
  6V \\
  V_2 \\
  3V \\
  12V \\
  I_2
\end{array}
\]
Solution

Step 1

Calculate $V_1$, assuming port 1 is the source & port 2 is open circuit

$$V_1 = (12 + 6)I_1 = 18I_1 \quad \& \quad V_2 = 6I_1$$

$$Z_{11} = \frac{V_1}{I_1} \bigg|_{I_2=0} = 18 \Omega \quad \quad Z_{21} = \frac{V_2}{I_1} \bigg|_{I_2=0} = 6 \Omega$$

Similarly, when a source is at port 2 and port 1 is open circuit

$$V_2 = (6 + 3)I_2 = 9I_2 \quad \& \quad V_1 = 6I_2$$

$$Z_{22} = \frac{V_2}{I_2} \bigg|_{I_1=0} = 9 \Omega \quad \quad Z_{12} = \frac{V_1}{I_2} \bigg|_{I_1=0} = 6 \Omega$$

$$\begin{bmatrix}
18 & 6 \\
6 & 9
\end{bmatrix}$$

Example

\[ \begin{array}{c}
I_1 \\
+ \\
V_1 \\
\downarrow \\
12\nu \\
\downarrow \\
+ \\
3\nu \\
V_2 \\
\downarrow \\
- \\
- \\
\downarrow \\
I_2 \\
+ \\
Solv \\
\begin{bmatrix}
12 & 0 \\
0 & 3
\end{bmatrix}
\end{array} \]
Admittance Parameters

\[ I_1 = Y_{11} V_1 + Y_{12} V_2 \]
\[ I_2 = Y_{21} V_1 + Y_{22} V_2 \]

\[
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix} =
\begin{bmatrix}
Y_{11} & Y_{12} \\
Y_{21} & Y_{22}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
\]

Example

\[ Y_{11} = \frac{I_1}{V_1} \Bigg|_{V_2 = 0} \]
\[ I_1 = 0.1 \left( 0.2 + 0.025 \right) \frac{V_1}{0.1 + 0.2 + 0.025} = \frac{0.0225}{0.325} V_1 \]
\[ \frac{Y_{11}}{Y_{12}} = \frac{I_1}{V_1} \Bigg|_{V_2 = 0} \frac{0.0225}{0.325} \]

If voltage across 0.25 is \( V_N \), then

\[ V_N = \frac{I_1}{0.2 + 0.025} = \frac{V_1}{3.25} \]
\[ I_{22} = -0.2V_N = \frac{-0.2}{3.25} V_1 \]

\[ Y_{21} = \frac{I_2}{V_1} \bigg|_{V_2=0} = \frac{-0.2}{3.25} = -0.06158 \]

\[ I_2 = \frac{0.025}{0.325} V_2 \]

\[ I_1 = \frac{-0.2}{3.25} V_2 \]

\[ Y_{12} = \frac{I_1}{V_2} \bigg|_{V_1=0} = \frac{-0.2}{3.25} = -0.06158 \]

\[ Y_{22} = \frac{I_2}{V_2} \bigg|_{V_1=0} = \frac{0.025}{0.325} = 0.07698 \]
Example 2:
Find admittance parameters of a transmission line of length $l$

\[ I_1 \xrightarrow{k} I_2 \]
\[ V_1 \xrightarrow{Z_0, \gamma} V_2 \]

Soln

→ This circuit is symmetrical because interchanging port 1 and port 2 does not affect it. Hence $Y_{22}$ must be equal to $Y_{11}$.

→ This circuit is reciprocal because if voltage $V$ at port 1 produces a short circuit current $I$ at port 2, voltage $V$ at port 2 will produce current $I$ at port 1. ($Y_{12} = Y_{21}$)

→ Assume a source is connected at port 1 when the other port has a short circuit. If $V_{in}$ is the incident voltage at port 1, it will appear as $V_{in}e^{-\gamma l}$ at port 2. Since the reflection coefficient of a short circuit is equal to $-1$, reflected voltage at this port is $180^\circ$ out of phase with incident voltage. Therefore reflected voltage at this port is $-V_{in}e^{-2\gamma l}$.
\[ V_1 = V_{in} - V_{in} e^{-2xl} \]

\[ V_2 = 0 \]

\[ \frac{I_1}{Z_o} = \frac{V_{in}}{Z_o} \left( 1 + e^{-2xl} \right) \]

\[ \frac{I_2}{Z_o} = -2V_{in} e^{-xl} \]

\[ Y_{11} = \left. \frac{I_1}{V_1} \right|_{V_2=0} = \frac{1}{Z_o \tanh \rho l} \]

\[ Y_{21} = \frac{-1}{Z_o \sinh \rho l} \]

\textbf{Transmission Parameters (elements of chain matrix)}

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}
\]

\[ B = \frac{V_1}{-I_2} \bigg|_{V_2=0} \]

\[ D = \frac{I_1}{-I_2} \bigg|_{V_2=0} \]

\[ A = \frac{V_1}{V_2} \bigg|_{I_2=0} \]

\[ C = \frac{I_1}{V_2} \bigg|_{I_2=0} \]
With a source at port 1 while port 2 has a short circuit:

\[ V_1 = \left( 1 + \frac{1}{1+j\omega} \right) I_1 = \frac{2+j\omega}{1+j\omega} I_1 \]

\[ I_2 = -\frac{j\omega}{j\omega + 1} I_1 = -\frac{1}{1+j\omega} I_1 \]

\[ B = \frac{V_1}{I_2} \bigg|_{V_2=0} = 2+j\omega \]

\[ D = \frac{I_1}{I_2} \bigg|_{V_2=0} = 1+j\omega \]

When source connected at port 1 while port 2 is open:

\[ V_1 = \left( 1 + \frac{1}{j\omega} \right) I_1 = \frac{1+j\omega}{j\omega} I_1 \quad \text{&} \quad V_2 = \frac{1}{j\omega} I_1 \]
\[ A = \frac{V_1}{V_2 \mid I_2 = 0} = 1 + j\omega \]

\[ C = \frac{I_1}{V_2 \mid I_2 = 0} = j\omega \]

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} =
\begin{bmatrix}
1 + j\omega & 2 + j\omega \\
j\omega & 1 + j\omega
\end{bmatrix}
\]

Example 2

Find transmission parameters of transmission line

\[ V_1 = V_{in} - V_{in} e^{-2\pi l} \]

\[ V_2 = 0 \]

\[ I_1 = \frac{V_{in}}{Z_0} (1 + e^{-2\pi l}) \]

\[ I_2 = \frac{2V_{in}}{Z_0} e^{-\pi l} \]

Therefore

\[ B = V_1 \mid \frac{-I_2}{V_2 = 0} = Z_0 \sin h \pi l \]

\[ D = \frac{I_1}{-I_2 \mid V_2 = 0} = \frac{1 + e^{-2\pi l}}{2e^{-\pi l}} = \cosh \pi l \]
\[ A = \frac{V_1}{V_2} \bigg|_{I_2=0} = \cosh r \lambda \]

\[ C = \frac{I_1}{V_2} \bigg|_{I_2=0} = \frac{1}{Z_0} \sinh r \lambda \]

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
\cosh r \lambda & Z_0 \sinh r \lambda \\
\frac{1}{Z_0} \sinh r \lambda & \cosh r \lambda
\end{bmatrix}
\]

**Scattering Parameters**

Z-parameters are useful in analyzing series circuits while Y-parameters are useful in analyzing parallel connected circuits. Similarly, transmission parameters are useful for chain or cascade circuits.

However, the characterization procedure of these parameters requires an open or short circuit at the other port. This extreme reflection makes it very difficult to determine the parameters of a network at radio and microwave frequencies. Therefore, a new representation based on traveling waves is defined. This is known as scattering matrix of the network. Elements of the matrix are called scattering parameters.
$b_1 = S_{11}a_1 + S_{12}a_2$

$b_2 = S_{21}a_1 + S_{22}a_2$

\[
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix} =
\begin{bmatrix}
  S_{11} & S_{12} \\
  S_{21} & S_{22}
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_2
\end{bmatrix}
\]

$S_{11} = \left. \frac{b_1}{a_1} \right|_{a_2 = 0}$

$S_{21} = \left. \frac{b_2}{a_1} \right|_{a_2 = 0}$

$S_{12} = \left. \frac{b_1}{a_2} \right|_{a_1 = 0}$

$S_{22} = \left. \frac{b_2}{a_2} \right|_{a_1 = 0}$
$S_{ii}$ is the reflection coefficient $P_i$ at $i^{th}$ port when the other port is matched terminated.

$S_{ij}$ is forward transmission coefficient of the $j^{th}$ port if $i$ is greater than $j$, whereas it represents the reverse transmission coefficient if $i$ is less than $j$ with the other port terminated by a matched load.

The steady state total voltage and current at the $i^{th}$ Port is given as:

\[ V_i = V_i^{\text{in}} + V_i^{\text{ref}} \]

\[ I_i = \frac{1}{Z_{0i}} (V_i^{\text{in}} - V_i^{\text{ref}}) \]

\[ V_i^{\text{in}} = \frac{1}{2} (V_i + Z_{0i} I_i) \]

\[ V_i^{\text{ref}} = \frac{1}{2} (V_i - Z_{0i} I_i) \]

Assuming both of the ports to be lossless so that $Z_{0i}$ is a real quantity, the average power incident at the $i^{th}$ port is:

\[ P_i^{\text{in}} = \frac{1}{2} \text{Re} \left[ V_i^{\text{in}} (I_i^{\text{in}})^* \right] = \frac{1}{2} \text{Re} \left[ V_i^{\text{in}} \left( \frac{V_i^{\text{in}}}{Z_{0i}} \right)^* \right] = \frac{1}{2} Z_{0i} |V_i^{\text{in}}|^2 \]
The average power reflected from the $i^{th}$ port is:

$$P_{i}^{\text{ref}} = \frac{1}{2} \text{Re} \left[ V_{i}^{\text{ref}} (I_{i}^{\text{ref}})^* \right] = \frac{1}{2} \text{Re} \left[ V_{i}^{\text{ref}} \left( \frac{V_{i}^{\text{ref}}}{Z_{0i}} \right)^* \right] = \frac{1}{2Z_{0i}} |V_{i}^{\text{ref}}|^2$$

$$a_i = \frac{V_{i}^{\text{in}}}{\sqrt{2Z_{0i}}} = \frac{1}{2} \left[ \frac{V_{i} + Z_{0i} I_{i}}{\sqrt{2Z_{0i}}} \right] = \frac{1}{2\sqrt{2}} \left( \frac{V_{i}}{\sqrt{Z_{0i}}} + \sqrt{Z_{0i}} I_{i} \right)$$

$$b_i = \frac{V_{i}^{\text{ref}}}{\sqrt{2Z_{0i}}} = \frac{1}{2} \left[ \frac{V_{i} - Z_{0i} I_{i}}{\sqrt{2Z_{0i}}} \right] = \frac{1}{2\sqrt{2}} \left( \frac{V_{i}}{\sqrt{Z_{0i}}} - \sqrt{Z_{0i}} I_{i} \right)$$

The $a_i$ and $b_i$ are defined in such a way that the squares of the magnitudes represent the power flowing in respective directions.

Power available from the source, $P_{\text{av}}$, at port 1 is

$$P_{\text{av}} = |a_1|^2$$

Power reflected from port 1, $P_{\text{ref}}$, is

$$P_{\text{ref}} = |b_1|^2$$

Power delivered to the port, $P_d$, is

$$P_d = P_{\text{av}} - P_{\text{ref}} = |a_1|^2 - |b_1|^2$$
Consider the following ckt:

\[ b_1 = S_{11}a_1 + S_{12}a_2 \quad - (1) \]

\[ b_2 = S_{21}a_1 + S_{22}a_2 \quad - (2) \]

\[ \Gamma_L = \frac{Z_L - Z_{02}}{Z_L + Z_{02}} = \frac{a_2}{b_2} \quad - (3) \]

\[ \Gamma_s = \frac{Z_s - Z_{01}}{Z_s + Z_{01}} = \frac{a_1}{b_1} \quad - (4) \]

The input and output reflection coefficients are

\[ \Pi_L = \frac{Z_L - Z_{01}}{Z_L + Z_{01}} = \frac{b_1}{a_1} \quad - (5) \]

\[ \Pi_s = \frac{Z_s - Z_{02}}{Z_s + Z_{02}} = \frac{b_2}{a_2} \quad - (6) \]

\[ \frac{b_1}{a_1} = \frac{Z_s - Z_{01}}{Z_s + Z_{01}} = S_{11} + S_{12} \quad - (7) \]

Substituting (5) in (2) after dividing (1) by \(a_1\). 
\[ \frac{b_2}{a_2} = S_{22} + S_{21} \frac{a_1}{a_2} = \frac{1}{\Pi_L} \Rightarrow \frac{a_1}{a_2} = \frac{1 - S_{22} \Pi_L}{S_{21} \Pi_L} \quad - (8) \]

From (7) & (8)

\[ \Gamma_1 = S_{11} + \frac{S_{12} S_{21} \Pi_L}{1 - S_{22} \Pi_L} \quad - (9) \]

If a matched load is terminating port 2, \( \Pi_L = 0 \)

we get

\[ \Gamma_1 = S_{11} \quad - (10) \]

From (2) and (4) we get

\[ \frac{b_2}{a_2} = \Gamma_2 = \frac{Z_2 - Z_{02}}{Z_2 + Z_{02}} = S_{22} + S_{21} \frac{a_1}{a_2} \quad - (11) \]

&

\[ \frac{b_1}{a_1} = S_{11} + S_{12} \frac{a_2}{a_1} = \frac{1}{\Pi_S} \Rightarrow \frac{a_2}{a_1} = \frac{1 - S_{11} \Pi_S}{S_{12} \Pi_S} \quad - (12) \]

From (11) & (12) we get

\[ \Gamma_2 = S_{22} + \frac{S_{21} S_{12} \Pi_S}{1 - S_{11} \Pi_S} \quad - (13) \]
If port 1 is matched and $P_s = 0$. Therefore

$$
\Gamma_2 = S_{22}
$$

$S_{ii}$ and $S_{22}$ can be found by evaluating the reflection coefficients at respective ports while the other port is matched terminated.

Let's determine $S_{21}$ and $S_{12}$

$$
S_{21} = \left. \frac{b_2}{a_1} \right|_{a_2 = 0}
$$

--- (14)---

Now $a_2$ can be found as

$$
a_2 = \frac{1}{2} \left[ \frac{V_2 + Z_2 I_2}{\sqrt{2Z_0}} \right]
$$

--- (15)---

For $a_2$ to 0, we get

$$
V_2 = -Z_2 I_2
$$

--- (16)---

Substituting (16) in the equation for $b_2$ we get

$$
b_2 = \frac{1}{2} \left( \frac{V_2 - Z_0 I_2}{\sqrt{2Z_0}} \right) = \frac{V_{s1}}{Z_0^{\frac{3}{2}}},
$$

--- (17)---
\[ S_{21} \mid_{a_2=0} = \frac{b_2}{a_1} = -\frac{\sqrt{2} Z_{02} I_2 / \sqrt{2}}{V_{S1} / 2 \sqrt{2} Z_{01}} = \frac{2V_2}{V_{S1} \sqrt{Z_{01}}} \sqrt{Z_{01} / Z_{02}} \]  \( \text{(17)} \)

\[ S_{12} \mid_{a_1=0} = \frac{b_1}{a_2} = \frac{2V_1 \cdot \sqrt{Z_{02}}}{V_{S2} \sqrt{Z_{01}}} \]  \( \text{(18)} \)

An analysis of S-parameters indicates that

\[ |S_{11}|^2 = \left| \frac{b_1}{a_1} \right|_{a_2=0} = \frac{P_{AVS} - P_d}{P_{AVS}} \]  \( \text{(19)} \)

\( P_{AVS} \) is power available at source.

\( P_d \) - Power delivered to port 1

These two powers will be equal if the source impedance is conjugate of \( Z_1 \); that is, the source is matched with port 1

\[ |S_{21}|^2 = \frac{Z_{02} (I_2 / V_2)^2}{(\frac{1}{4} Z_{01}) (V_{S1} / V_2)^2} = \frac{Z_{02} (I_2 / V_2)^2}{\frac{1}{2} \left[ (\frac{1}{4} Z_{01}) (V_{S1} / V_2)^2 \right]} = \frac{P_{AVN}}{P_{AVS}} \]  \( \text{(20)} \)

\( P_{AVN} \) - Power available at port 2

It will be equal to power delivered to a load that is matched to the port.
The power ratio of (2) may be called the transducer power gain.

It can be found that $|S_{22}|^2$ represents the ratio of power reflected from port 2 to power available from the source at port 2, while port 1 is terminated by a matched load $Z_s$ and $|S_{12}|^2$ represents a reverse transducer power gain.