Problem 1. Apply the orthogonal principle to prove that the best linear minimum mean square error (LMMSE) estimator of $X$ given $Y$ is

$$\hat{X}(Y) = EX + \text{Cov}(X,Y)\text{Cov}(Y,Y)^{-1}(Y - EY),$$

and the corresponding MSE is

$$\text{MSE} = \text{Var}(X) - \text{Cov}(X,Y)\text{Cov}(Y,Y)^{-1}\text{Cov}(Y,X).$$

Here we assume that $X$ is a scalar random variable and $Y$ is a random vector of dimension $n$. Hint: The LMMSE estimator has the form $a + c^H Y$, where $a$ is a scalar, and $c$ is a deterministic vector of dimension $n$. Apply the orthogonal principle $E(e \cdot 1) = 0$ and $E(e \cdot Y^H) = 0$, where $e = X - (a + c^H Y)$ to solve for $a$ and $c$.

Problem 2.

Consider the channel model $y = hx + w$, where $y$, $h$, and $w$ are $n \times 1$ vectors, and $w$ is a complex Gaussian random vector with distribution $CN(0, \sigma_w^2 I_n)$. Assume that $x$ is a scalar random variable such that $Ex = 0$ and $\sigma_x^2 = E(x^2)$. The channel gain $h$ is assumed to be fixed and known to the receiver. Apply the two formula from Problem 1 to find the best LMMSE estimator of $x$ given $y$. Verify that the LMMSE solution is a scaled version of the matched filter solution. Compute the MSE of the LMMSE estimator.

Problem 3.

Consider $r = b_0u_0 + Z$, where $r$, $u_0$, and $Z$ are $n \times 1$ vectors. The noise vector $Z$ has a Gaussian distribution $N(0, \Sigma_Z)$. Assume that $b_0$ and $Z$ are independent.

1. Show that for any linear detector of the form $c^H r$, the corresponding mean square error $\text{MSE} = \sigma_b^2(c^H u_0 - 1)^2 + c^H \Sigma_Z c$.

2. Use the orthogonal principle to show that the minimum MSE achieved by the optimal LMMSE receiver $c_M$ equals $\sigma_b^2(1 - c_M^H u_0)$.

3. Use the results of (1) and (2) to show that for the optimal LMMSE receiver $c_M$, we have

$$\text{SINR} = \frac{\sigma_b^2}{\text{MSE}} - 1 = \frac{c_M^H u_0}{1 - c_M^H u_0}.$$

Problem 4.

Consider the following channel model

$$r = b_0 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + b_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + n,$$

where $b_i, i = 0, 1, 2$ are BPSK symbols, and the noise vector $n$ has a Gaussian distribution of $N(0, \sigma^2 I_4)$.

In the following, we consider different linear detectors of the form $c^H r$ to detect symbol $b_0$. First, we assume that $\sigma^2 = 0.1$. 

1
(1) Find the matched filter solution $c_{MF}$. Compute the corresponding signal-to-interference plus noise ratio (SINR) at the correlator output.

(2) Find the zero-forcing solution $c_{MF}^{(1)}$ to knock out symbols $b_1$ and $b_2$. Compute the corresponding SINR at the correlator output. Also, compute the error probability of this zero-forcing detector.

(3) Find the zero-forcing solution $c_{MF}^{(2)}$ to knock out symbol $b_1$. Compute the corresponding SINR at the correlator output. Also, compute the exact error probability of this zero-forcing detector. Here we assume that $b_i, i = 0, 1, 2$ are equally likely to be 1 or −1.

(4) Find the LMMSE solution $c_M$. Compute the corresponding SINR at the correlator output. Also, compute the MSE.

(5) Based on results from (1)-(4), verify the fact that $c_M$ performs more closely to $c_{ZF}$ than $c_{MF}$ at high SNR.

(6) Repeat (1)-(4) for $\sigma^2 = 10$.

(7) Based on results from (1)-(4), verify that $c_M$ performs more closely to $c_{MF}$ than $c_{ZF}$ at low SNR.