EX: Find a radially symmetric joint probability density function, \( f(x, y) \), for which \( X \) and \( Y \) are independent. That is, find an \( f(x, y) \) that may be written as a function of \( x^2 + y^2 \). Hint: consider what type of function turns multiplication into addition.

SOL'N: The function that turns multiplication into addition is the exponential:

\[
 e^x e^y = e^{x+y}
\]

We replace \( x \) and \( y \) with \( x^2 \) and \( y^2 \) to obtain a function of \( x^2 + y^2 \). We then add a minus sign to obtain functions that have finite area over the interval \((-\infty, \infty)\). Finally, we need a normalizing constant to make the area of each function equal to one so we have valid probability density functions.

Calculating the normalizing constant by calculating the integral of \( e^{-x^2} \) directly requires advanced complex analysis. Instead, we observe that we have a gaussian density function with \( \sigma^2 = \frac{1}{2} \):

\[
 f_X(x) = \frac{1}{\sqrt{2\pi} \sigma^2} e^{-x^2/2\sigma^2}
\]

Thus, we have the following \( f_X(x) \) and \( f_Y(y) \):

\[
 f_X(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \quad \quad f_Y(y) = \frac{1}{\sqrt{\pi}} e^{-y^2}
\]

Since \( X \) and \( Y \) are independent, the probability density function \( f(x, y) \) is the product of \( f_X(x) \) and \( f_Y(y) \):

\[
 f(x, y) = \frac{1}{\pi} e^{-(x^2+y^2)}
\]

This definition holds for all real \( x \) and \( y \). The plots below show the shape of this 2-dimensional gaussian (but with \( \sigma^2 = 1 \)). Note the circular symmetry in the contour plot.

What is remarkable about the circularly symmetric gaussian is that, since \( x \) and \( Y \) are independent, every cross section must have the same shape after being scaled vertically to achieve an area equal to one. A cylindrically-shaped \( f(x, y) \) would have cross sections of different widths, for example.
Standard 2-D gaussian probability density function

Standard 2-D gaussian probability density function: Topographic Map