Notes for Chapter 3 : Waveguides

For rectangular waveguides

TE Waves
pp. 106 – 110

$E_z = 0$

$$\left(\nabla^2 + k_c^2\right) H_z = 0 \quad (3.73)\ \text{p.}106$$

$$H_z = A_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}\, e^{-j\beta z} \quad (3.81)$$

From Eqs. 3.5 a – d, for TE/TM waves

$$\vec{H}_t = \frac{-j\beta}{k_c^2} \nabla_t H_z - \frac{j\omega\epsilon}{k_c^2} \cancel{\hat{z} \times \nabla_t E_z}^{\,0} \quad (1)$$

$$\vec{E}_t = \frac{j\omega\mu}{k_c^2} \hat{z} \times \nabla_t H_z - \frac{j\beta}{k_c^2} \cancel{\nabla_t E_z}^{\,0} \quad (2)$$

$$Z_{TE} = \frac{|E_t|}{|H_t|} = \frac{\omega\mu}{\beta} \quad (3.86)$$

$$\frac{\omega_c^2}{c_e^2} \rightarrow k_c^2 = k^2 - \beta^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad (3.83),\ \text{also } (3.102)$$

z y x

for TE₁₀ mode (the lowest order/cutoff frequency mode)

$m = 1,\ n = 0$

$$H_z = A_{10} \cos\frac{\pi x}{a}\, e^{-j\beta z} \quad (3.89a)$$

from Eqns. (1), (2)

$$E_y = \frac{j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial x} = \frac{-j\omega\mu a A_{10}}{\pi} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z} \quad (3.89b)$$

$$E_x = \frac{-j\omega\mu}{k_c^2} \frac{\partial H_z}{\partial y} = 0$$

$$H_x = \frac{-j\beta}{k_c^2} \frac{\partial H_z}{\partial x} = \frac{j\beta a}{\pi} A_{10} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z} \quad (3.89c)$$

$$H_y = E_x = E_z = 0 \quad (3.89d)$$

TM Waves
pp. 111 – 115

$H_z = 0$

$$\left(\nabla^2 + k_c^2\right) E_z = 0 \quad (3.97)$$

$$E_z = B_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, e^{-j\beta z} \quad (3.100)$$

From Eqs 3.5 a – d, for ~~TE~~/TM waves ($H_z = 0$)

$$\rightarrow \vec{H}_t = \frac{-j\omega\epsilon}{k_c^2} \hat{z} \times \nabla_t E_z \quad (3)$$

$$\rightarrow \vec{E}_t = \frac{-j\beta}{k_c^2} \nabla_t E_z \quad (4)$$

$$Z_{TM} = \left|\frac{\vec{E}_t}{\vec{H}_t}\right| = \frac{\beta}{\omega\epsilon} \quad (3.104)$$

for the lowest cutoff frequency TM₁₁ mode

$H_z = 0$

$m = 1,\ n = 1$

$$E_z = B_{11} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}\, e^{-j\beta z}$$

We can write the remaining components E_x, E_y, H_x, H_y from Eqns. (3), (4)

Some useful relationships for rectangular Waveguides

	TE	TM
Cutoff frequencies	$f_{c_{mn}} = \frac{c_\epsilon}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \frac{n^2}{b^2}}$ (3.84)	$\frac{c_\epsilon}{2}\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$
Lowest cut off frequency mode	TE_{10} ; $f_c\Big\|_{TE_{10}} = \frac{c_\epsilon}{2a}$ (3.85)	TM_{11} ; $f_c\Big\|_{TM_{11}} = \frac{c_\epsilon}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$ (3.103)
Propagation constant $\beta = \frac{\omega}{v_p}$	$\frac{\sqrt{\omega^2 - \omega_c^2}}{c_\epsilon}$ (3.83)	$\frac{\sqrt{\omega^2 - \omega_c^2}}{c_\epsilon}$ (3.102)
Wave impedance	$Z_{TE} = \frac{\omega\mu}{\beta} = \frac{\eta_\epsilon}{\sqrt{1-(f_c^2/f^2)}}$ (3.86)	$Z_{TM} = \frac{\beta}{\omega\epsilon} = \eta_\epsilon\sqrt{1-(f_c^2/f^2)}$ (3.104)
Guide Wavelength λ_g	$\lambda_g = \frac{\lambda_\epsilon}{\sqrt{1 - f_c^2/f^2}} = \frac{\lambda_0/\sqrt{\epsilon_r}}{\sqrt{1-(f_c^2/f^2)}}$	$\lambda_g = \frac{\lambda_0/\sqrt{\epsilon_r}}{\sqrt{1-(f_c^2/f^2)}}$
Expression for power flow in the WG	$P_{10} = \frac{ab}{4Z_{TE}}\left(E_y\Big\|_{x=a/2}\right)^2 = \frac{ab}{4Z_{TE}}(E_{max})^2$ (3.92)	
Attenuation Constant due to conductor losses $\alpha_c\Big\|_{\text{nepers/m}}$	$\alpha_c\Big\|_{TE_{10}} = \frac{R_s}{b\eta_\epsilon}\frac{f_c}{f}\frac{\left[\left(\frac{f^2}{f_c^2}\right) + \frac{2b}{a}\right]}{\sqrt{\left(\frac{f^2}{f_c^2}\right) - 1}}$ (3.96), with $\frac{1}{\sigma\delta_s} \rightarrow R_s$	$\alpha_c\Big\|_{TM_{mn}} = \frac{2R_s}{b\eta_\epsilon\sqrt{1-\left(\frac{f_{mn}^2}{f^2}\right)}}\frac{m^2\frac{b^3}{a^3} + n^2}{m^2\frac{b^2}{a^2} + n^2}$
	Multiply by 8.686 **to get** α_c in dB/m	
due to dielectric losses $\alpha_d\Big\|_{\text{nepers/m}}$	$\frac{\epsilon''/\epsilon'}{2\beta}\;\frac{\omega^2}{c_\epsilon^2}$	$\frac{\epsilon''/\epsilon'}{2\beta}\;\frac{\omega^2}{c_\epsilon^2}$
Maximum power handling capability or power that will cause air breakdown	$3.6\; a\Big\|_{inch}\; b\Big\|_{inch}\sqrt{1-(f_c^2/f^2)}$ MW	—

Table 2.6. Salient features of some commercially available rectangular waveguides.

EIA Designation WR( )	Inside Dimensions Inches	Cutoff Frequency c/2a GHz	Recommended Frequency Range for TE_{10} Mode GHz	Theoretical CW Power Rating for Lowest to Highest Frequency MW
2300	23.000-11.500	0.256	0.32-0.49	153.0-212.0
2100	21.000-10.500	0.281	0.35-0.53	120.0-173.0
1800	18.000-9.000	0.328	0.41-0.625	93.4-131.9
1500	15.000-7.500	0.393	0.49-0.75	67.6-93.3
1150	11.500-5.750	0.513	0.64-0.96	35.0-53.8
975	9.750-4.875	0.605	0.75-1.12	27.0-38.5
770	7.700-3.850	0.766	0.96-1.45	17.2-24.1
650	6.500-3.250	0.908	1.12-1.70	11.9-17.2
510	5.100-2.550	1.157	1.45-2.20	7.5-10.7
430	4.300-2.150	1.372	1.70-2.60	5.2-7.5
340	3.400-1.700	1.736	2.20-3.30	3.1-4.5
284	2.840-1.340	2.078	2.60-3.95	2.2-3.2
229	2.290-1.145	2.577	3.30-4.90	1.6-2.2
187	1.872-0.872	3.152	3.95-5.85	1.4-2.0
159	1.590-0.795	3.711	4.90-7.05	0.79-1.0
137	1.372-0.622	4.301	5.85-8.20	0.56-0.71
112	1.122-0.497	5.259	7.05-10.00	0.35-0.46
90	0.900-0.400	6.557	8.20-12.40	0.20-0.29
75	0.750-0.375	7.868	10.00-15.00	0.17-0.23
62	0.622-0.311	9.486	12.40-18.00	0.12-0.16
51	0.510-0.255	11.574	15.00-22.00	0.080-0.107
42	0.420-0.170	14.047	18.00-26.50	0.043-0.058
34	0.340-0.170	17.328	22.00-33.00	0.034-0.048
28	0.280-0.140	21.081	26.50-40.00	0.022-0.031
22	0.224-0.112	26.342	33.00-50.00	0.014-0.020
19	0.188-0.094	31.357	40.00-60.00	0.011-0.015
15	0.148-0.074	39.863	50.00-75.00	0.0063-0.0090
12	0.122-0.061	48.350	60.00-90.00	0.0042-0.0060
10	0.100-0.050	59.010	75.00-110.00	0.0030-0.0041
8	0.080-0.040	73.840	90.00-140.00	0.0018-0.0026
7	0.065-0.0325	90.840	110.00-170.00	0.0012-0.0017
5	0.051-0.0255	115.750	140.00-220.00	0.00071-0.00107
4	0.043-0.0215	137.520	170.00-260.00	0.00052-0.00075
3	0.034-0.0170	173.280	220.00-325.00	0.00035-0.00047

3.4 Circular Waveguide

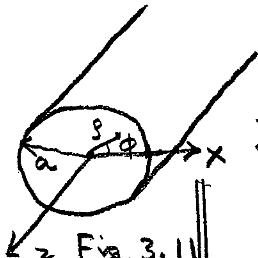


Fig. 3.11

TE modes pp. 118–121

$E_z = 0$

$$(\nabla^2 + k_c^2) H_z = \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\right) H_z = 0 \quad 3.112$$

$$H_z(\rho, \phi) = (A \sin n\phi + B \cos n\phi) J_n(k_c \rho)\, e^{-j\beta z} \quad (3.121)$$

Similar to Eqs. (1), (2) on p. 20-1

$$\vec{H}_t = \frac{-j\beta}{k_c^2} \nabla_t H_z \quad (5)$$

$$\vec{E}_t = \frac{j\omega\mu}{k_c^2} \hat{z} \times \nabla_t H_z \quad (6)$$

From p. 682 of the Text, for the transverse part of ∇ (in cylindrical coordinates)

$$\nabla_t = \hat{\rho}\frac{\partial}{\partial \rho} + \frac{\hat{\phi}}{\rho}\frac{\partial}{\partial \phi} \quad (9)$$

$$Z_{TE} = \frac{|\vec{E}_t|}{|\vec{H}_t|} = \frac{\omega\mu}{\beta} = \frac{\eta_\epsilon}{\sqrt{1 - \frac{f_c^2}{f^2}}} \quad (3.129)$$

From Eqs (6) and (9)

$$E_\phi = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial \rho} = \frac{j\omega\mu}{k_c}(A \sin n\phi + B \cos n\phi) J_n'(k_c \rho)\, e^{-j\beta z} \quad (3.123)$$

E_ϕ is tangential to the metallic wall at $\rho = a$

$E_\phi = 0$ @ $\rho = a$

$$J_n'(k_c a) = 0 \quad (3.124)$$

The mth root of J_n' is called

$p_{nm}' \left(= k_c a = \frac{\omega_c|_{TE_{nm}}}{c_\epsilon} a\right)$

These roots are tabulated in Table 3.3 on p. 119 of the Text

From Eqns. 3.121, (5) and (6), the rest of the field components i.e. E_ϕ, E_ρ, H_ϕ, H_ρ can be written. These are given as Eqs. 3.130 a–f in the Text

TM modes pp. 122–126

$H_z = 0$

$$(\nabla^2 + k_c^2) E_z = 0 \quad (3.134)$$

$$E_z = (A \sin n\phi + B \cos n\phi) J_n(k_c \rho)\, e^{-j\beta z} \quad (3.135)$$

From Eqs. (3), (4) on p. 20-1

$$\vec{H}_t = -\frac{j\omega\epsilon}{k_c^2} \hat{z} \times \nabla_t E_z \quad (7)$$

$$\vec{E}_t = \frac{-j\beta}{k_c^2} \nabla_t E_z \quad (8)$$

$$Z_{TM} = \left|\frac{\vec{E}_t}{\vec{H}_t}\right| = \frac{\beta}{\omega\epsilon} = \eta_\epsilon \sqrt{1 - \frac{f_c^2}{f^2}} \quad (3.142)$$

E_z is tangential to the metallic walls

$E_z = 0$ @ $\rho = a$

$$J_n(k_c a) = J_n(p_{nm}) = 0 \quad 3.138$$

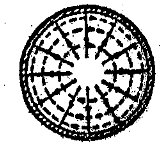
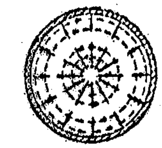
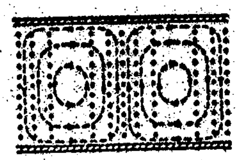
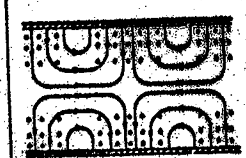
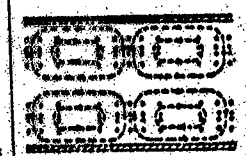
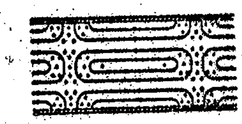
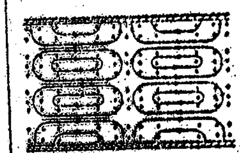
Values of p_{nm} are tabulated in Table 3.4 on p. 122 of the Text

From Eqs. 3.135 and (7), (8) the rest of the field components can be written. The expressions for E_ρ, E_ϕ, H_ρ, H_ϕ are given as Eqns. 3.141 a–d of the Text.

Some useful relationships for circular Waveguides

	TE	TM
Cutoff frequencies	$f_{c_{nm}} = \dfrac{p'_{nm} c_\epsilon}{2\pi a}$ (3.127) p'_{nm} given in Table 3.3 p.135	$f_{c_{nm}} = \dfrac{p_{nm} c_\epsilon}{2\pi a}$ (3.140) p_{nm} given in Table 3.4 p.137
Lowest cutoff frequency mode	TE_{11} ; $f_c\Big\vert_{TE_{11}} = \dfrac{1.841\, c_\epsilon}{2\pi a}$	TM_{01} ; $f_c\Big\vert_{TM_{01}} = \dfrac{2.405\, c_\epsilon}{2\pi a}$
Propagation constant β	$\dfrac{\sqrt{\omega^2 - \omega_c^2}}{c_\epsilon}$ (3.126)	$\dfrac{\sqrt{\omega^2 - \omega_c^2}}{c_\epsilon}$ (3.139)
Recommended band for single mode (TE_{11}) propagation	(1.15) $\dfrac{1.841 c_\epsilon}{2\pi a} \leq f \leq 0.95\, \dfrac{3.054\, c_\epsilon}{2\pi a}$ $1.15\, f_c\Big\vert_{TE_{11}} \leq f \leq 0.95\, f_c\Big\vert_{TE_{21}}$	
Guide wavelength $\lambda_g = \dfrac{2\pi}{\beta}$	$\lambda_g = \dfrac{\lambda_\epsilon}{\sqrt{1-(f_c^2/f^2)}}$	$\dfrac{\lambda_\epsilon}{\sqrt{1-(f_c^2/f^2)}}$
Expression for power flow in the WG	$\dfrac{\pi \omega \mu_0 \vert A\vert^2 \beta}{4 k_c^4}(p'^2_{11}-1) J_1^2(k_c a)$ (3.131)	$\dfrac{\pi \omega \epsilon \beta a^2 \vert A\vert^2}{(p_{01})^2} \dfrac{a^2}{2} J_1'^2(k_c a)$
Attenuation constant due to conductor losses $\alpha_c\Big\vert_{\text{nepers/m}}$	$\alpha_c\Big\vert_{TE_{11}} = \dfrac{R_s}{a \eta_\epsilon} \dfrac{1}{\sqrt{1-(f_c^2/f^2)}} \left\{ \dfrac{f_c^2}{f^2} + \dfrac{1}{p'^2_{11}-1} \right\}$ (3.133)	$\dfrac{R_s}{a \eta_\epsilon} \dfrac{1}{\sqrt{1-(f_c^2/f^2)}}$
due to dielectric losses α_d	$\dfrac{\epsilon''/\epsilon'}{2\beta} \dfrac{\omega^2}{c_\epsilon^2}$ (3.28)	$\dfrac{\epsilon''/\epsilon'}{2\beta} \dfrac{\omega^2}{c_\epsilon^2}$ (3.28)
Maximum power handling capability or power that will cause air breakdown	$2.7\,(2a)^2 \sqrt{1-(f_c^2/f^2)}$ MW where a is in inches.	—

Some lower order modes of a circular Waveguide
(Also See p. 125 Text Fig. 3.14)

Wave Type	TE_{11}	TM_{01}	TE_{01}	TM_{11}	TM_{02}
Field distributions in cross-sectional plane, at plane of maximum transverse fields	 Distributions below along this plane			 Distributions below along this plane	
Field distributions along guide					
Field components present	$\boxed{H_z, H_r, H_\theta, E_r, E_\theta}$	E_z, E_r, H_θ	H_z, H_r, E_θ	$E_z, E_r, E_\theta, H_r, H_\theta$	E_z, E_r, H_θ
f_c	$\dfrac{1.84\, c_\varepsilon}{2\pi b}$ a	$\dfrac{2.405\, c_\varepsilon}{2\pi b}$ a	$\dfrac{3.83\, c_\varepsilon}{2\pi b}$	$\dfrac{3.83\, c_\varepsilon}{2\pi b}$	$\dfrac{5.52\, c_\varepsilon}{2\pi b}$
Attenuation due to imperfect conductors * nepers/m	$\dfrac{R_s}{b\eta_\varepsilon} \dfrac{1}{\sqrt{1-(f_c/f)^2}} \left[\left(\dfrac{f_c}{f}\right)^2 + 0.420\right]$	$\dfrac{R_s}{b\eta_\varepsilon} \dfrac{1}{\sqrt{1-(f_c/f)^2}}$	$\dfrac{R_s}{b\eta_\varepsilon} \dfrac{(f_c/f)^2}{\sqrt{1-(f_c/f)^2}}$	$\dfrac{R_s}{b\eta_\varepsilon} \dfrac{1}{\sqrt{1-(f_c/f)^2}}$	$\dfrac{R_s}{b\eta_\varepsilon} \dfrac{1}{\sqrt{1-(f_c/f)^2}}$

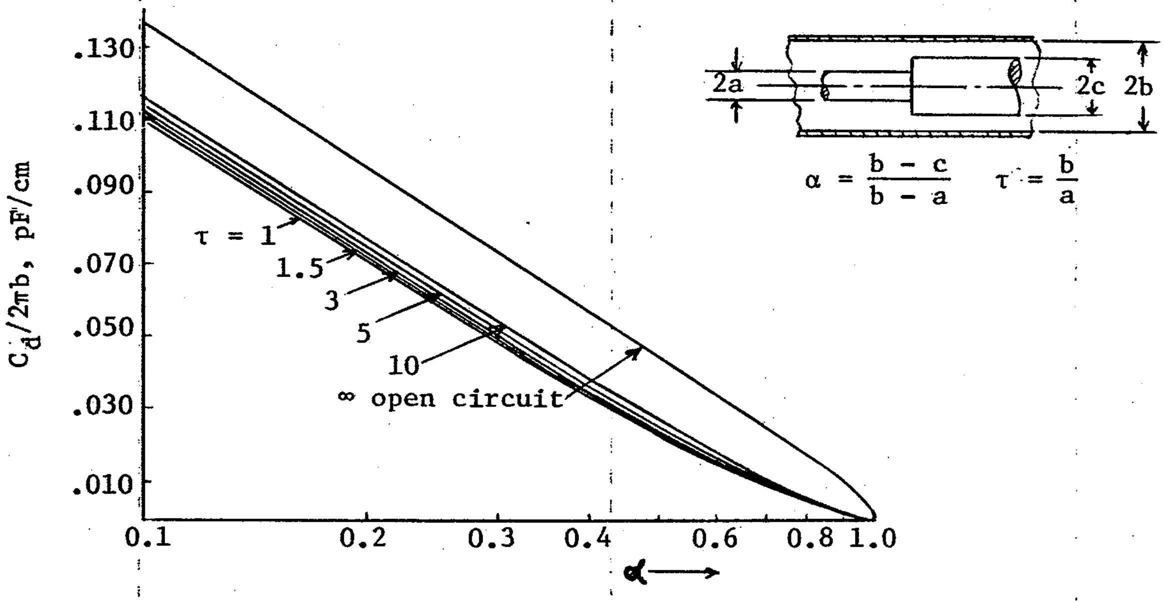
* $R_s = 1/\sigma\delta$ $\eta_\varepsilon = \sqrt{\mu/\varepsilon} = 377/\sqrt{\varepsilon_r}$ ohms

Fig. 4.19. Field patterns of some of the lowest frequency modes of a circular waveguide. [Source: S. Ramo [1].]

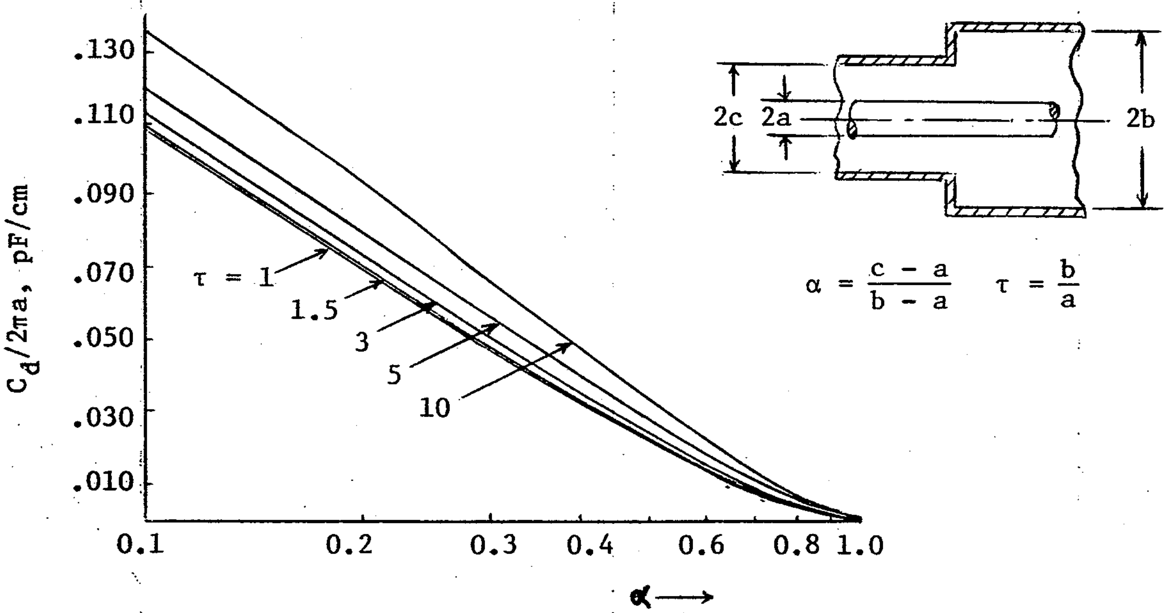
(31)

Table 2.8. Some standard circular waveguides.
(All dimensions are in inches.)

EIA Designation	Inside Dimensions			Recommended Frequency Range TE_{11} Mode, GHz
	Diameter	Tolerance + or −	Roundness Tolerance	
WC 992	9.915	.010	.010	0.803–1.10
WC 847	8.470	.008	.008	0.939–1.29
WC 724	7.235	.007	.007	1.10–1.51
WC 618	6.181	.006	.006	1.29–1.76
WC 528	5.280	.005	.005	1.51–2.07
WC 451	4.511	.005	.005	1.76–2.42
WC 385	3.853	.004	.005	2.07–2.83
WC 329	3.292	.003	.003	2.42–3.31
WC 281	2.812	.003	.003	2.83–3.88
WC 240	2.403	.0025	.002	3.31–4.54
WC 205	2.047	.002	.002	3.89–5.33
WC 175	1.750	.0015	.0015	4.54–6.23
WC 150	1.500	.0015	.0015	5.30–7.27
WC 128	1.281	.0013	.0013	6.21–8.51
WC 109	1.094	.001	.0011	7.27–9.97
WC 94	0.938	.0009	.0009	8.49–11.6
WC 80	0.797	.0008	.0008	9.97–13.7
WC 69	0.688	.0007	.0007	11.6–15.9
WC 59	0.594	.0006	.0006	13.4–18.4
WC 50	0.500	.0005	.0005	15.9–21.8
WC 44	0.438	.00045	.0004	18.2–24.9
WC 38	0.375	.00038	.0004	21.2–29.1
WC 33	0.328	.00033	.0003	24.3–33.2
WC 28	0.281	.00028	.0001	28.3–38.8
WC 25	0.250	.00025	.0001	31.8–43.6
WC 22	0.219	.00025	.0001	36.4–49.8
WC 19	0.188	.00025	.00007	42.4–58.1
WC 17	0.172	.00025	.00007	46.3–63.5
WC 14	0.141	.00025	.00005	56.6–77.5
WC 13	0.125	.00025	.00005	63.5–87.2
WC 11	0.109	.00025	.00005	72.7–99.7
WC 9	0.094	.00025	.00005	84.8–116

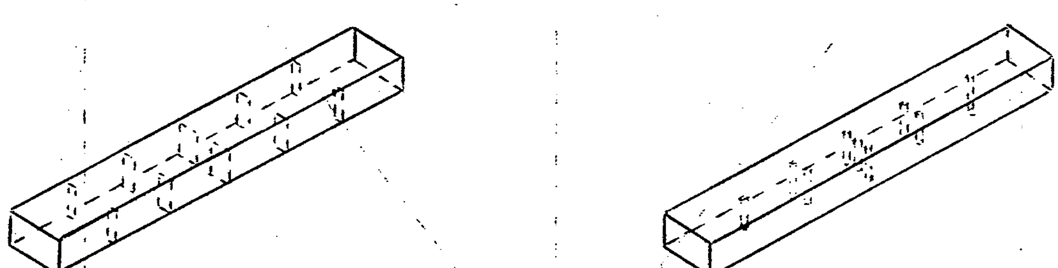


(a) Step on inner conductor.



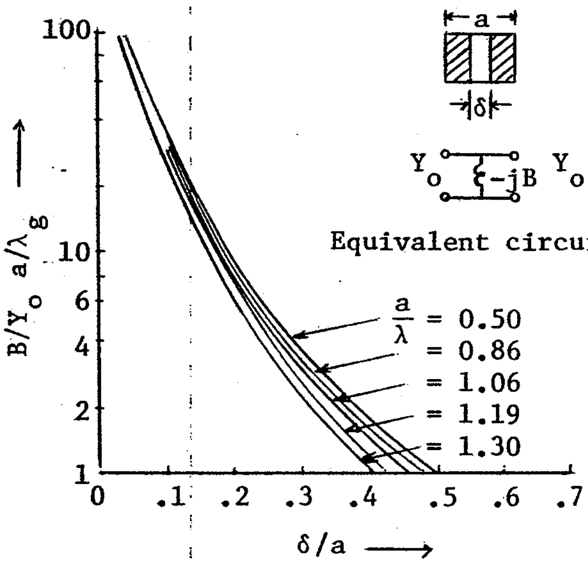
(b) Step on outer conductor.

Fig. 8.21. Discontinuity capacitances for steps in a coaxial line. [Source: G. L. Matthaei, *et al.* [7] and J. R. Whinnery and H. W. Jamieson [8].]

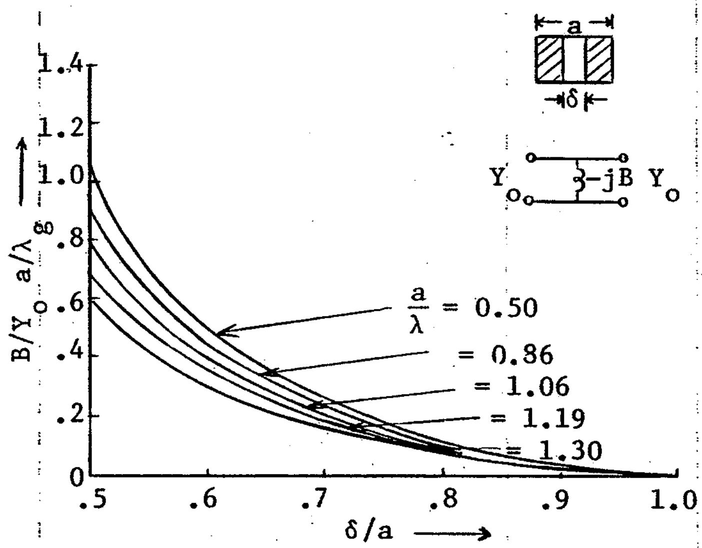


(a) (b)

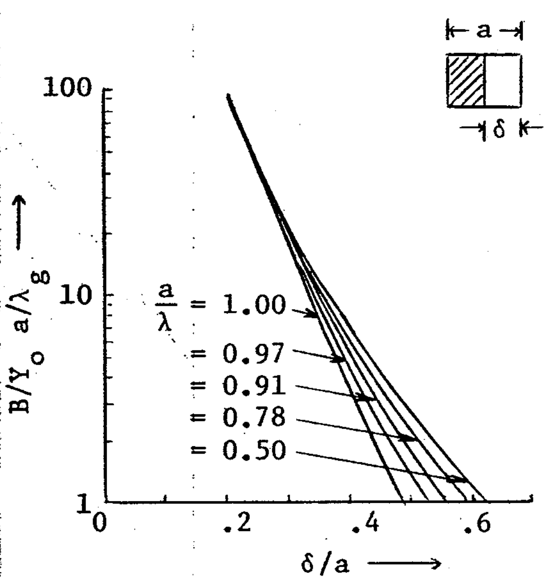
Fig. 8.22. Some typical waveguide bandpass filters.



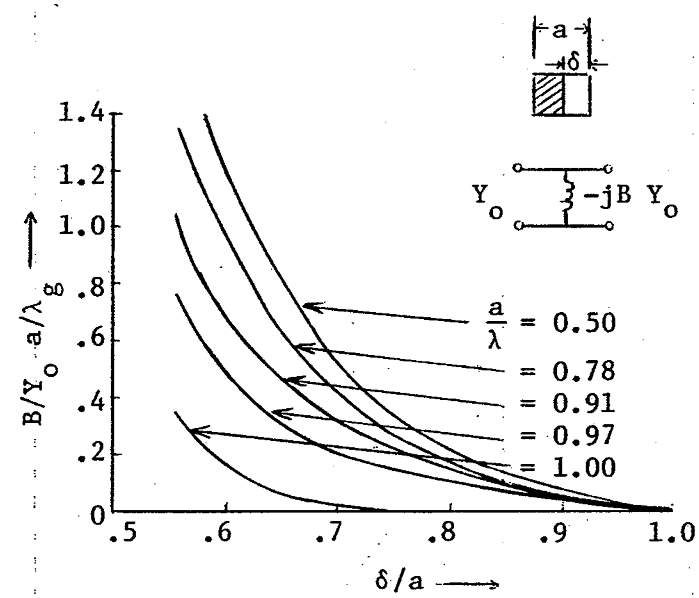
(a) Symmetrical inductive iris ($\delta/a \le 0.5$).



(b) Symmetrical inductive iris ($\delta/a \ge 0.5$).

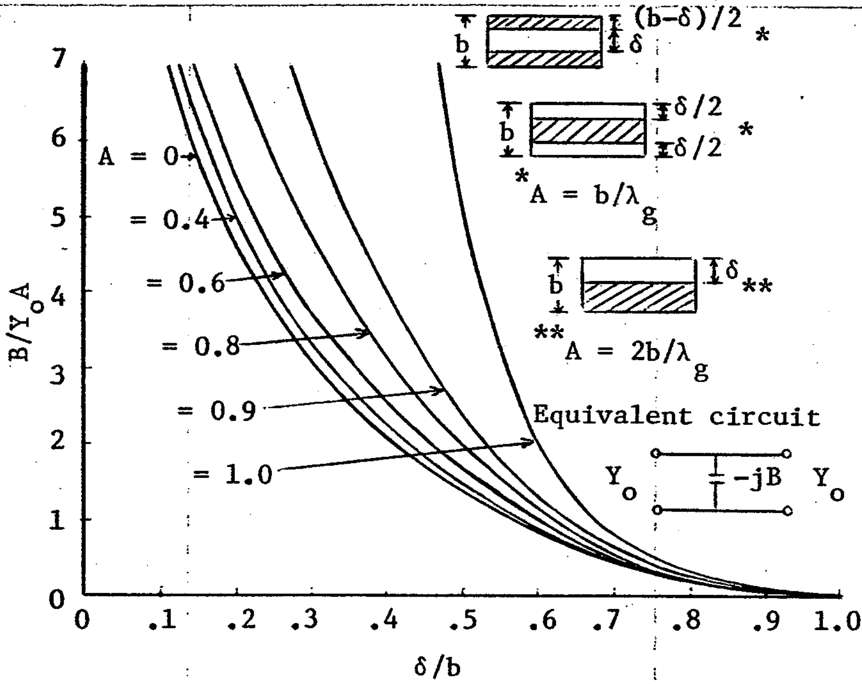


(c) Asymmetrical inductive iris ($\delta/a < 0.6$).

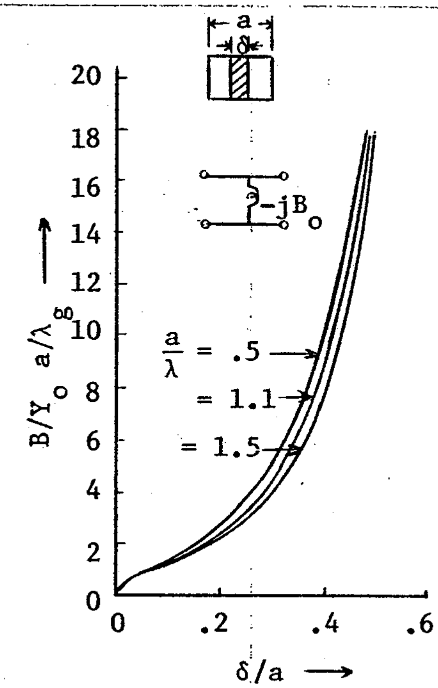


(d) Asymmetrical inductive iris ($\delta/a > 0.5$).

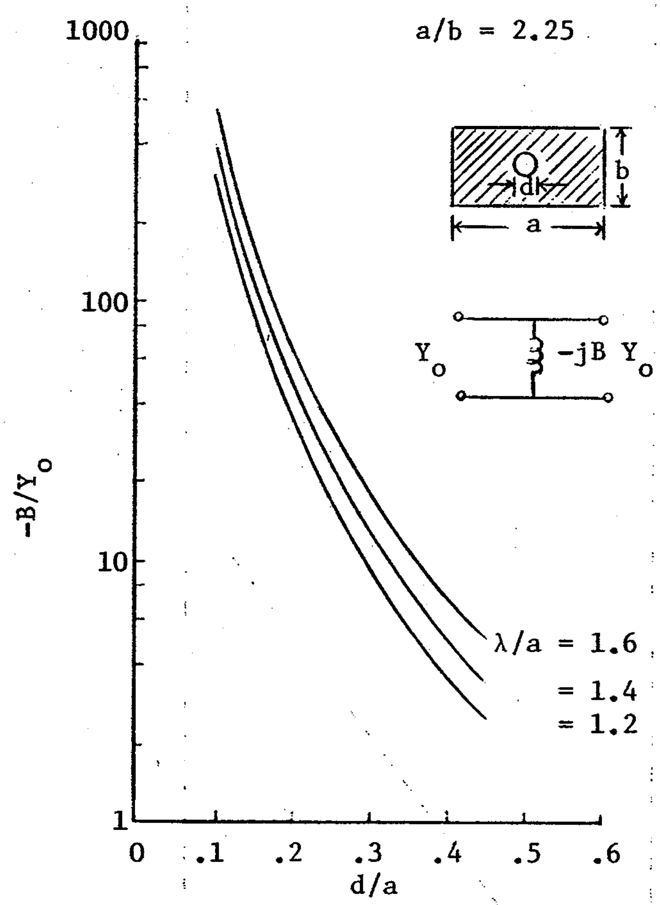
Fig. 8.23. Susceptance of discontinuities in a waveguide. [Source: G. L. Matthaei, *et al.* [7].]



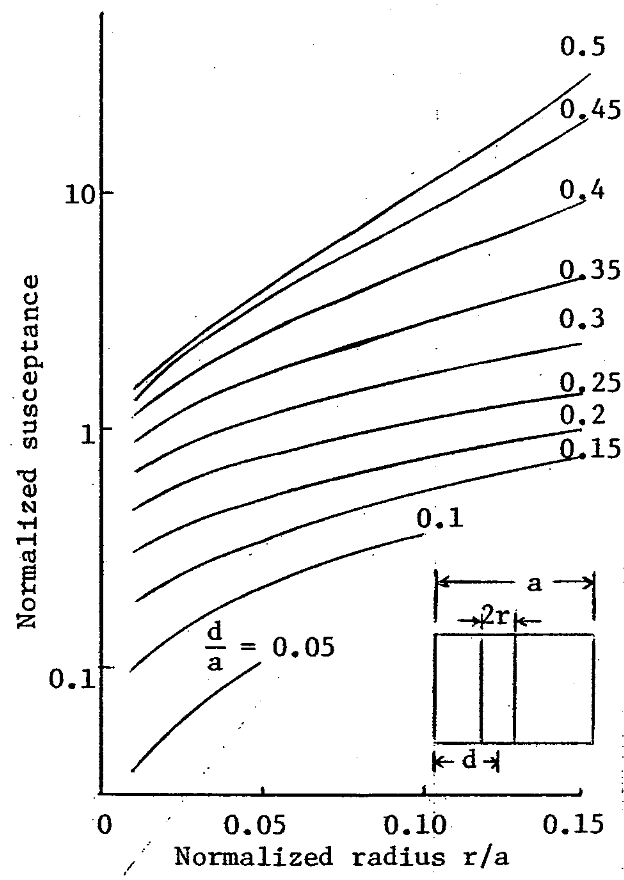
(e) Capacitive irises.



(f) Centered thin vane.

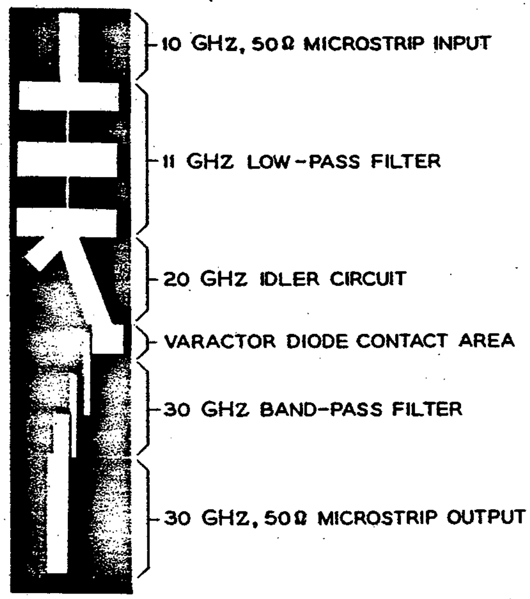


(g) Hole in iris.

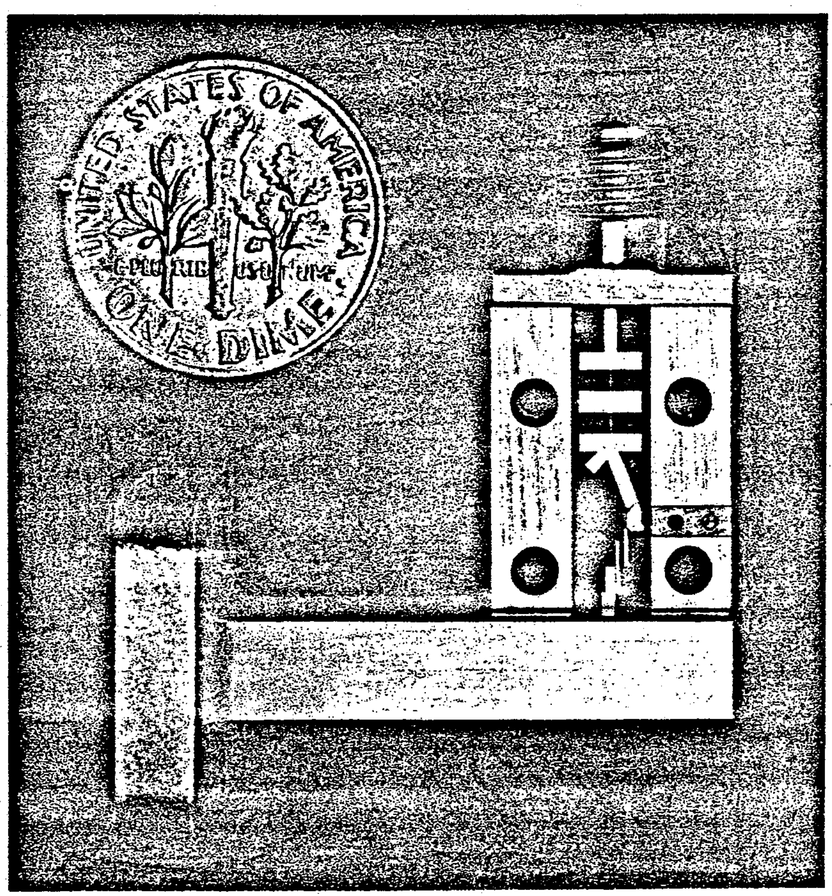


(h) A single post in a waveguide ($\lambda_g/a = 1.6$).

Fig. 8.23, continued.



(a) Conductor pattern on silica substrate



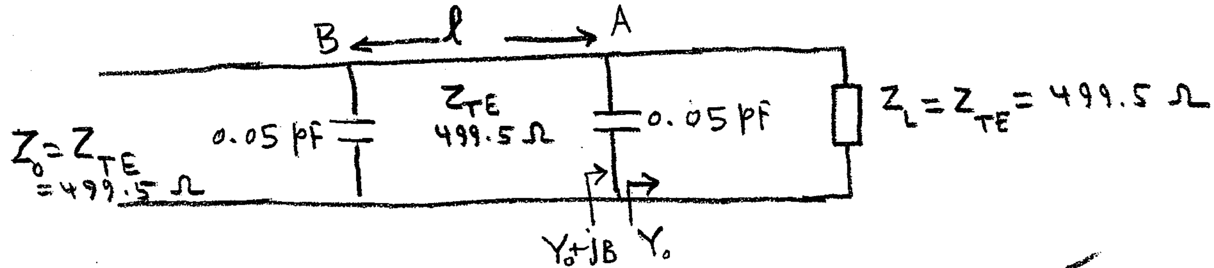
(b) The physical layout including the coax to microstrip and microstrip to WR 28 waveguide transitions

Fig. 8.24. An integrated circuit 10-30 GHz harmonic generator. [Source: M. V. Schneider and W. W. Snell, Jr. [9].]

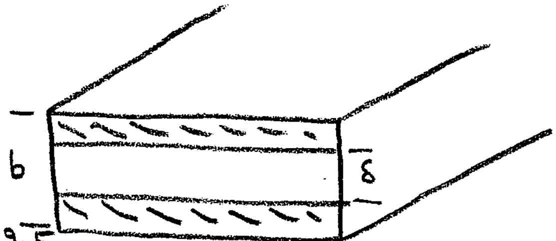
Ex. Design of a Waveguide Bandpass Filter (Similar to HW Prob. 3)

WR 90 waveguide @ 10 GHz

(p. 36



In order to get an effective iris capacitance of 0.05 pF at 10 GHz



A Capacitive iris or diaphragm

$$\frac{B}{Y_o A} = \frac{\omega C}{Y_o A} = \frac{2\pi \times 10^{10} \times 0.05 \times 10^{-12} \times 499.5}{\frac{0.4 \times 2.54}{3.975}}$$

b/λ_g

$$= 6.14$$

$$A = \frac{b}{\lambda_g} = \frac{0.4 \times 2.54}{3.975} = 0.2556$$

From graph on capacitive irises (p.26 of Notes)

$$\boxed{\frac{\delta}{b} = 0.13}$$

normalized y for plane A

$$y_A = \frac{Y_o + jB}{Y_o} = 1 + j\frac{B}{Y_o}$$

$$= 1 + j\, 6.14 \times 0.255$$

$$= \boxed{1 + j\, 1.57}$$

From the attached Smith Chart, for matching

$$l/\lambda_g = (0.322 - 0.178) = 0.144$$

$$l = 0.144 \times 3.975 = 0.57 \text{ cm}$$

Multiple section filters can also be made using this procedure.

For an inductive iris $\frac{B}{Y_o} = -1.57$; $\frac{Z_o}{\omega L} = 1.57$; $L = \frac{499.5}{2\pi \times 10^{10} \times 1.57}$

$$= 5.06 \text{ nH}$$

For a symmetrical inductive iris

$$\frac{B}{Y_o}\frac{a}{\lambda_g} = 1.57 \times \frac{0.9 \times 2.54}{3.975} = 0.903$$

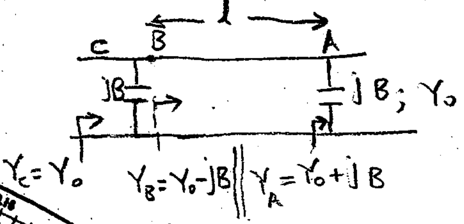
$$\frac{a}{\lambda_o} = \frac{0.9 \times 2.54}{3.0} = 0.762$$

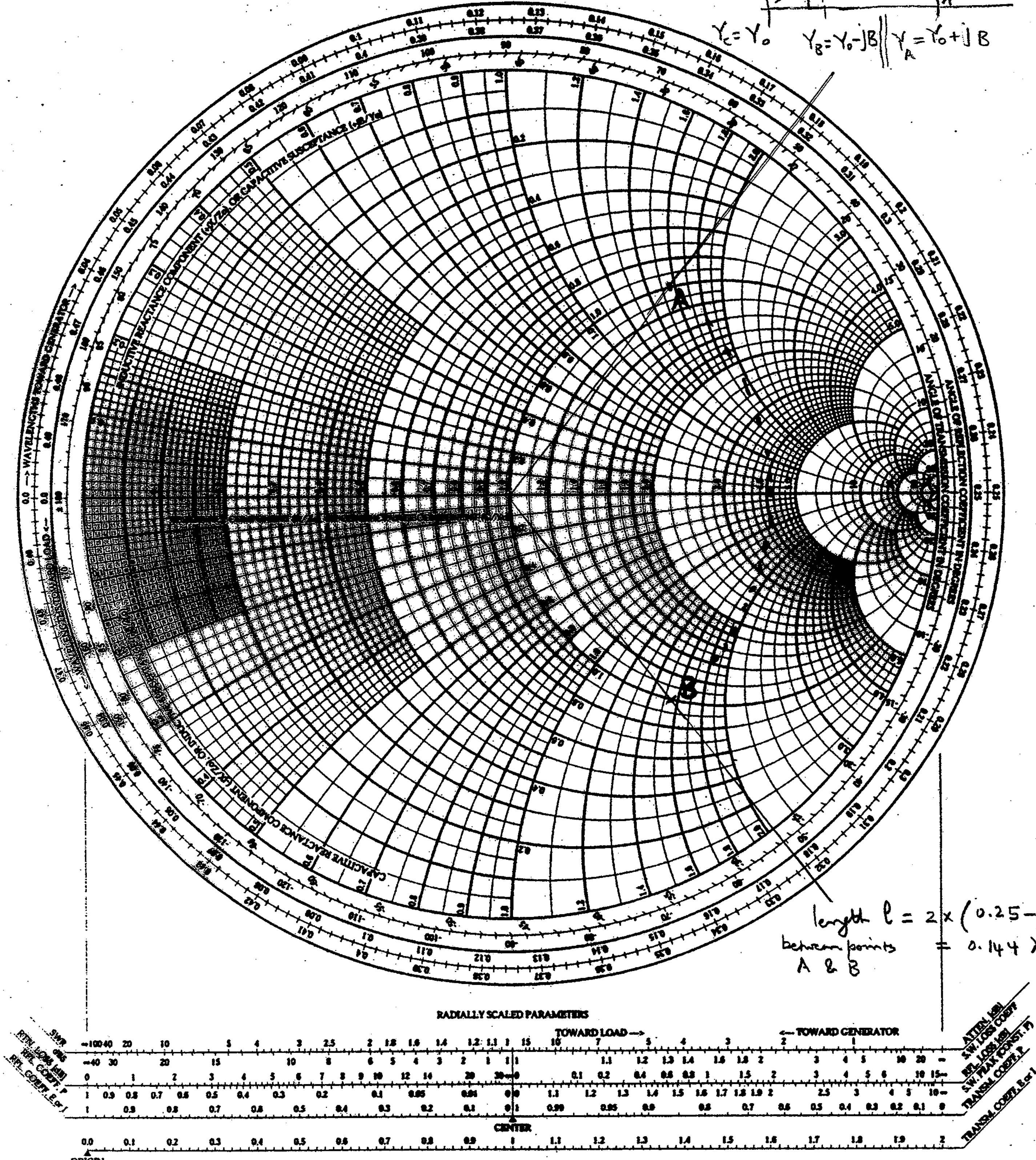
From graph on p.25 $\rightarrow$ $\frac{\delta}{a} \sim 0.51$

for $\frac{a}{\lambda} = 0.762$

The Complete Smith Chart

Black Magic Design





length $l = 2 \times (0.25 - 0.178)$
between points A & B $= 0.144\,\lambda_g$

Chapter 3 – Section 3.1 (pp. 92 – 97)

From

O. P. Gandhi – Microwave Engineering and Applications, Pergamon Press, 1980. (see also pp. 104–110 Pozar Text)

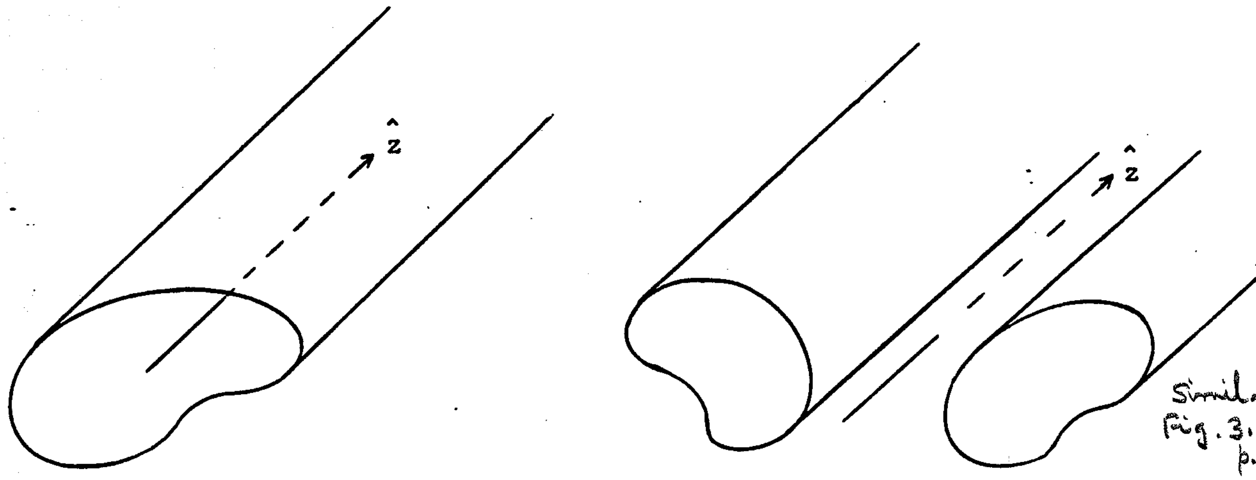
p. 105 Text by Pozar

2.2 Unified Theory -- Wave Propagation in a General "Cylindrical" Waveguide

A general [1] cylindrical waveguiding system is an arrangement of conductor/s, hollow or solid, which have the unchanging cross section as a function of z, which is the direction of propagation of energy. Some typical schematics of a general cylindrical waveguide are illustrated in Fig. 2.1.

From the definition of the general cylindrical waveguide, it can be seen that all waveguiding systems fit this definition. A circular waveguide, a rectangular waveguide, elliptical and triangular waveguides, coaxial line, parallel plate striplines, etc., have the same cross-sectional shape at various values of z and are consequently general cylindrical waveguides.

In most textbooks the above waveguiding systems are treated separately, equations of wave propagation solved, and the field patterns of various possible modes are derived. In this text we shall derive certain general unifying relationships in a general cylindrical waveguiding system since these are not that complicated



Similar to Fig. 3.1 Text p. 105

(a) A single hollow conducting tube of arbitrary cross section that maintains the same shape at all values of z.

(b) Two or more solid conductors of arbitrary cross section parallel to one another either side by side or one enclosing the others.

Fig. 2.1. Some typical general cylindrical waveguiding systems.

either to derive mathematically or to comprehend physically. The advantage we shall gain by following this approach, unlike that of most senior level texts, is that these general relationships, once derived, would be applied to the various systems to derive the possible modes without need for repetitious rederivation for each geometry.

A *waveguide mode* is a unique arrangement of the electric and magnetic fields propagating in the z-direction that satisfies *all* the Maxwell's equations (Appendix A) and the boundary conditions imposed by the geometry of the conductor/s of the transmission system (line). In other words, the various waveguide modes are different possible propagating solutions of the Maxwell's equations subject to the boundary conditions of the system. Having obtained these solutions by solving the boundary value problem, it will be necessary to launch the so-derived unique arrangement of the electric and magnetic fields on the system to assure the excitation of a single waveguide mode.

The various waveguide modes possible in waveguides may be categorized in one of the four possible categories:

a. Transverse ElectroMagnetic mode or TEM mode.
   In this mode both the electric and magnetic fields are purely transverse to the direction of propagation and consequently have no z-directed components; i.e., both E_z and H_z are zero.
b. Transverse Electric mode or TE mode.
   In this mode only the electric field is purely transverse to the

direction of propagation and the magnetic field is not purely transverse; i.e., $E_z = 0$, $H_z \neq 0$.

c. Transverse Magnetic mode or TM mode.
In this mode only the magnetic field is purely transverse to the direction of propagation while the electric field is not purely transverse and therefore would have a component in the direction of propagation; i.e., $H_z = 0$, $E_z \neq 0$.

d. Hybrid modes or HE modes.
These modes have neither the electric nor the magnetic field purely transverse to the direction of propagation; i.e., both $\vec{E}$ and $\vec{H}$ have components along the direction of propagation; $E_z \neq 0$, $H_z \neq 0$.

2.3 Field Relationships in a General "Cylindrical" Waveguide

The schematic of a general "cylindrical" waveguide is shown in Fig. 2.1. From Maxwell's equations (derived in Appendix A):

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -j\omega\mu\vec{H} \tag{2.1}$$
1.41a Text

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = j\omega\varepsilon\vec{E} \tag{2.2}$$
1.41 b Text

$$\nabla \cdot \vec{B} = 0 \tag{2.3}$$

$$\nabla \cdot \vec{D} = 0 \tag{2.4}$$

Taking the curl of Eq. 2.1, we obtain the wave equation for $\vec{E}$ in a general coordinate system:

$$\nabla \times \nabla \times \vec{E} = -j\omega\mu\ \nabla \times \vec{H} = \omega^2\mu\varepsilon\vec{E} \equiv k_\varepsilon^2\vec{E} \tag{2.5}$$

where k_ε denotes $\omega\sqrt{\mu\varepsilon}$ which is ω/c_ε with $c_\varepsilon = 1/\sqrt{\mu\varepsilon}$ being the velocity of light in the dielectric medium filling the waveguide. k_ε, in other words, is the propagation constant of electromagnetic waves in the filler medium of infinite cross section.

From vector calculus (see Appendix B) it can be shown that

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2\vec{E} \tag{2.6}$$

in the Cartesian system or for the z-component of $\vec{E}$ in the cylindrical system. The exception for the z-component is because it is indistinguishable between the Cartesian system (where Eq. 2.6 is valid) and the cylindrical system.

Since $\nabla \cdot \vec{E} \equiv 0$ from Eq. 2.4, the wave equation for the z-component E_z for our general waveguide can be written as

$\omega^2 \mu \epsilon$

$$\nabla^2 E_z + k_\varepsilon^2 E_z = 0 \tag{2.7}$$ 1.42 Text

By starting with the curl of Eq. 2.2 and proceeding in a similar manner as above, the wave equation for the z-component of magnetic field can be obtained as

$$\nabla^2 H_z + k_\varepsilon^2 H_z = 0 \tag{2.8}$$ 1.43 Text

Solving the wave equations (2.7) and (2.8) subject to the boundary conditions of the waveguide, we can, in principle, obtain the solutions for E_z and H_z. It will now be our attempt to find the transverse (remaining) components of the electric and magnetic fields $\vec{E}_t$ and $\vec{H}_t$ so that $\vec{E}$ and $\vec{H}$ are completely known. It should be mentioned at this stage that *$\vec{E}_t$ and $\vec{H}_t$ represent two components each* in the transverse plane. See Eq 3.1 a 3.1 b p. 105 Text

The operator del ∇ similarly is also written as a sum of the z-directed part and the transverse part; i.e.,

$$\vec{\nabla} \equiv \vec{\nabla}_z + \vec{\nabla}_t \tag{2.9}$$

Once again $\vec{\nabla}_t$ represents two components as may be seen by the illustrative examples of the Cartesian and cylindrical coordinate systems. In Cartesian coordinates,

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \equiv \vec{\nabla}_t + \vec{\nabla}_z \tag{2.10}$$

where

$$\vec{\nabla}_t = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} \quad \text{and} \quad \vec{\nabla}_z \equiv \hat{z}\frac{\partial}{\partial z} \tag{2.11}$$

In cylindrical coordinates,

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{z}\frac{\partial}{\partial z} \equiv \vec{\nabla}_t + \vec{\nabla}_z \tag{2.12}$$

and

$$\vec{\nabla}_t = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} \tag{2.13}$$

Equating the transverse components on both sides of Eq. 2.1:

$$\begin{aligned} \vec{\nabla} \times \vec{E}\Big|_t &= \left(\vec{\nabla}_z + \vec{\nabla}_t\right) \times \left(\vec{E}_z + \vec{E}_t\right)\Big|_t \\ &= \vec{\nabla}_z \times \vec{E}_t + \vec{\nabla}_t \times \vec{E}_z = -j\omega\mu\vec{H}_t \end{aligned} \tag{2.14}$$

Similarly, from Eq. 2.2,

$$\vec{\nabla}_z \times \vec{H}_t + \vec{\nabla}_t \times \vec{H}_z = j\omega\varepsilon\vec{E}_t \tag{2.15}$$

Cross multiplying both sides of Eq. 2.15 by $\vec{\nabla}_z$, we obtain:

$$\vec{\nabla}_z \times \left(\vec{\nabla}_z \times \vec{H}_t\right) + \vec{\nabla}_z \times \left(\vec{\nabla}_t \times \vec{H}_z\right) = j\omega\varepsilon\, \vec{\nabla}_z \times \vec{E}_t \tag{2.16}$$

Making use of the vector relationship (Appendix B)

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - (\vec{A} \cdot \vec{B})\vec{C} \tag{2.17}$$

Eqn. B.5 p 7 in Text

we can write

$$\vec{\nabla}_z \times \left(\vec{\nabla}_z \times \vec{H}_t\right) = \vec{\nabla}_z \cancelto{0}{\left(\vec{\nabla}_z \cdot \vec{H}_t\right)} - \left(\vec{\nabla}_z \cdot \vec{\nabla}_z\right) \vec{H}_t \tag{2.18}$$

$$\vec{\nabla}_z \times \left(\vec{\nabla}_t \times \vec{H}_z\right) = \vec{\nabla}_t \left(\vec{\nabla}_z \cdot \vec{H}_z\right) - \cancelto{0}{\left(\vec{\nabla}_z \cdot \vec{\nabla}_t\right)} \vec{H}_z \tag{2.19}$$

Since in waveguide transmission we are interested in solutions capable of propagation along the length of the guide, we are searching for solutions of $\vec{E}$ and $\vec{H}$ that vary as $e^{-j\beta z}$, in which case $\vec{\nabla}_z \equiv \hat{z}\, \partial/\partial z \equiv -j\beta\hat{z}$ and Eq. 2.16 can be written as:

$$\beta^2 \vec{H}_t - j\beta\, \vec{\nabla}_t H_z = j\omega\varepsilon\, \vec{\nabla}_z \times \vec{E}_t \tag{2.20}$$

Upon substituting for $\vec{\nabla}_z \times \vec{E}_t \equiv -j\omega\mu\vec{H}_t - \vec{\nabla}_t \times \vec{E}_z$ from Eq. 2.14, the right side of Eq. 2.20 is written as:

$$j\omega\varepsilon \left[-j\omega\mu\vec{H}_t - \vec{\nabla}_t \times \vec{E}_z\right] = k_\varepsilon^2\vec{H}_t - j\omega\varepsilon\, \vec{\nabla}_t \times \vec{E}_z$$

$$= k_\varepsilon^2\vec{H}_t + j\omega\varepsilon\, \hat{z} \times \vec{\nabla}_t E_z \tag{2.21}$$

Upon rearranging terms, Eq. 2.20 is written as:

$$\boxed{\left(k_\varepsilon^2 - \beta^2\right) \vec{H}_t = -j\omega\varepsilon\, \hat{z} \times \vec{\nabla}_t E_z - j\beta\, \vec{\nabla}_t H_z} \tag{2.22}$$

An equation for $\vec{E}_t$ is similarly derived by cross multiplying Eq. 2.14 with $\vec{\nabla}_z$:

$$\boxed{\left(k_\varepsilon^2 - \beta^2\right) \vec{E}_t = +j\omega\mu\, \hat{z} \times \vec{\nabla}_t H_z - j\beta\vec{\nabla}_t E_z} \tag{2.23}$$

Equations 2.22 and 2.23 are important general equations that allow the evaluation of the transverse components $\vec{E}_t$ and $\vec{H}_t$ once E_z and H_z are known (from the solutions of Eq. 2.7 and Eq. 2.8 subject to the boundary conditions).

Transverse ElectroMagnetic or TEM Modes ($E_z = 0$, $H_z = 0$)

Equations 2.22 and 2.23 are not too useful for TEM modes since for these modes the right sides of the equations are zero, and that means that either $\vec{E}_t = 0$, $\vec{H}_t = 0$ which would imply zero fields, which is a trivial solution, or that $\beta = k_\varepsilon$. The latter solution is a powerful relationship, which means that for TEM modes the waves propagate at the velocity corresponding to that of unbounded electromagnetic waves in the filler medium.

Other than this important result, Eqs. 2.22 and 2.23 are incapable of allowing us the calculation of $\vec{E}_t$ and $\vec{H}_t$ for which an alternative approach will be developed in the next Section 2.4.

Transverse Electric or TE Modes ($E_z = 0$, $H_z \neq 0$)

For these modes $E_z = 0$; H_z is obtained from the solution of Eq. 2.8, and the general equations (2.22) and (2.23) can be simplified to:

$k_c^2 = k_\varepsilon^2 - \beta^2$

$H_x = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial x}$ 3.19 a

$H_y = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial y}$ 3.19 b

$$\boxed{\vec{H}_t = -\frac{j\beta}{k_\varepsilon^2 - \beta^2}\vec{\nabla}_t H_z} \qquad (2.24)$$

$E_x = \frac{\omega\mu}{\beta}H_y = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}$ 3.19 c

$E_y = -\frac{\omega\mu}{\beta}H_x = +\frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x}$ 3.19 d

$$\boxed{\vec{E}_t = \frac{j\omega\mu}{k_\varepsilon^2 - \beta^2}\hat{z} \times \vec{\nabla}_t H_z = \frac{\omega\mu}{\beta}\vec{H}_t \times \hat{z}} \qquad (2.25)$$

From Eqs. 2.24 and 2.25, it can be seen that the transverse fields $\vec{E}_t$ and $\vec{H}_t$ are at right angles to one another at all points in the transverse plane. Furthermore $\vec{E}_t$, $\vec{H}_t$, and $\hat{z}$ are oriented such that they form a right-handed coordinate system $\left(\vec{E}_t \| \left(\vec{H}_t \times \hat{z}\right), \hat{z} \| \left(\vec{E}_t \times \vec{H}_t\right)\text{, etc.}\right)$. Also, the magnitude of $\vec{E}_t$ at all points is $(\omega\mu/\beta)$ times that of $\vec{H}_t$. The ratio of the magnitudes of $\vec{E}_t$ and $\vec{H}_t$, having the dimensions of the impedance, is called the *wave impedance*.

The TE modes wave impedance Z_{TE} is therefore:

$$\boxed{Z_{TE} = \frac{\omega\mu}{\beta} = 377\frac{\lambda_g}{\lambda_o} \text{ ohms}} \qquad (2.26)$$

3.22 p. 109 Text

where λ_g $(= 2\pi/\beta)$ and λ_o are the guide and free-space wavelengths, respectively.

A physical interpretation of the *guide wavelength* λ_g $(= 2\pi/\beta)$ is that at any given instant of time, the fields are 2π out of phase and consequently repetitive at points that are axially separated by λ_g.

Since at the metallic waveguide walls only tangential components of magnetic field and normal component of electric field can exist, that means that at the waveguide walls the normal component of Eq. 2.24 should be zero. Denoting by $\hat{n}$ the unit vector normal to the waveguide wall pointing out of the metal,

$$\vec{H}_n = \hat{n} \times \vec{H}_t = \frac{-j\beta}{k_\varepsilon^2 - \beta^2}\, \hat{n} \cdot \vec{\nabla}_t H_z = -\frac{j\beta}{k_\varepsilon^2 - \beta^2} \frac{\partial H_z}{\partial n} = 0 \tag{2.27}$$

at the waveguide walls, or simply

$$\partial H_z/\partial n = 0 \tag{2.28}$$

at the waveguide walls. This is an important boundary condition that is needed to obtain the solution for H_z from Eq. 2.8. After solving for H_z, $\vec{H}_t$ and $\vec{E}_t$ are obtained from Eqs. 2.24 and 2.25 and the fields are then completely known for a given TE mode.

Transverse Magnetic or TM Modes ($H_z = 0$, $E_z \neq 0$)

For these modes $H_z = 0$ and E_z is obtained from the solution of Eq. 2.7 subject to the boundary condition that E_z being tangential electric field is zero at the metallic waveguide walls. Having obtained a solution for E_z in this manner, the remaining field components are obtained from the simplified versions of the general equations (2.22) and (2.23); i.e.,

$E_x = -\frac{j\beta}{k_c^2} \frac{\partial E_z}{\partial x}$ 3.23 (c)

$$\boxed{\vec{E}_t = \frac{-j\beta}{k_\varepsilon^2 - \beta^2} \vec{\nabla}_t E_z} \tag{2.29}$$

$E_y = -\frac{j\beta}{k_c^2} \frac{\partial E_z}{\partial y}$ 3.23(d)

$H_x = \frac{j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial y}$ 3.23(a)

$$\boxed{\vec{H}_t = \frac{-j\omega\varepsilon}{k_\varepsilon^2 - \beta^2} \hat{z} \times \vec{\nabla}_t E_z = \frac{\omega\varepsilon}{\beta} \hat{z} \times \vec{E}_t} \tag{2.30}$$

$H_y = -\frac{j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial x}$ 3.23 (b)

As in the case of the TE modes, for TM modes too, the fields $\vec{E}_t$ and $\vec{H}_t$ are at right angles at all points in the transverse plane, with $\vec{E}_t$, $\vec{H}_t$, and $\hat{z}$ forming a right-handed coordinate system. The $\vec{E}_t$ and $\vec{H}_t$ amplitudes are related by the factor $\beta/\omega\varepsilon$ and the *wave impedance* Z_{TM}, being the ratio of the magnitudes of $\vec{E}_t$ and $\vec{H}_t$, is given by:

$$\boxed{Z_{TM} = \frac{\beta}{\omega\varepsilon} = 377 \frac{\lambda_o}{\lambda_g} \text{ ohms}} \tag{2.31}$$

3.26 Text

HE or Hybrid Modes

For these modes the complete equations (2.22) and (2.23) are to be used.

2.4 TEM Modes in Waveguiding Systems

For such modes, by definition, the electric and magnetic fields are purely transverse to the direction of propagation. In Section 2.3 we showed that such modes propagate with a propagation constant k_ε, i.e., with the phase velocity corresponding to the velocity of unbounded electromagnetic waves c_ε in the filler medium. An interesting feature of these waves is that they can be excited down to zero frequency while, as will be shown later, the other modes, namely TE, TM, or HE, can only be excited above certain frequencies, called the cutoff frequencies.

It will be shown in this section that two or more conductors are, however, needed in the transmission line to excite TEM modes which therefore would make it impossible to excite TEM modes in single conductor systems such as the rectangular and circular waveguides and hollow pipes of other cross-sectional shape such as elliptical, wedge-shaped, etc.

Since the electric and magnetic fields $\vec{E}_t$ and $\vec{H}_t$ cannot be derived from the general equations (2.22) and (2.23), we start from the basic Maxwell's equations (from Appendix A):

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} \tag{2.32}$$

$$\nabla \times \vec{E} = -j\omega\mu_o\vec{H} \tag{2.33}$$

$$\nabla \cdot \vec{B} = 0 \tag{2.34}$$

$$\nabla \cdot \vec{D} = 0 \tag{2.35}$$

From Eq. 2.34, $\vec{B}$ can be expressed as curl of a vector $\vec{A}$ which is called the magnetic vector potential.

$$\vec{B} = \vec{\nabla} \times \vec{A} = \left(\vec{\nabla}_t + \vec{\nabla}_z\right) \times \left(\vec{A}_t + \vec{A}_z\right) \tag{2.36}$$

From Eq. 2.36 the z-component and the transverse components of $\vec{B}$ can be written:

$$\vec{B}_z = \vec{\nabla}_t \times \vec{A}_t \tag{2.37}$$

$$\vec{B}_t = \vec{\nabla}_z \times \vec{A}_t + \vec{\nabla}_t \times \vec{A}_z \tag{2.38}$$

Now, since for TEM fields, $\vec{B}_z \equiv 0$, that means from Eq. 2.37 that $\vec{A}_t \equiv 0$; in other words, for TEM fields, the magnetic vector potential $\vec{A}$ is purely z-directed, $\vec{A} \equiv \vec{A}_z$.

From Eq. 2.33, we can write,

$$\nabla \times \left(\vec{E} + j\omega\vec{A}_z\right) = 0 \tag{2.39}$$

The term $\vec{E} + j\omega\vec{A}_z$ within the parentheses can, therefore, be written as the gradient of a scalar function ψ since $\nabla \times (-\nabla\psi) \equiv 0$. Hence,

$$\vec{E} = -\vec{\nabla}\psi - j\omega\vec{A}_z \tag{2.40}$$

Now since $\vec{E}_z \equiv 0$ for TEM waves, from Eq. 2.40,

$$-\vec{\nabla}_z\psi - j\omega\vec{A}_z = jk_\varepsilon\psi\hat{z} - j\omega A_z\hat{z} = 0 \tag{2.41}$$

or

$$\psi = \frac{\omega}{k_\varepsilon} A_z \tag{2.42}$$

In terms of ψ therefore, from Eq. 2.40,

$$\boxed{\vec{E}_t = -\vec{\nabla}_t\psi} \tag{2.43}$$

from Eq. 2.38,

$$\boxed{\vec{H}_t = \frac{1}{\mu_o}\vec{\nabla}_t \times \vec{A}_z = -\frac{k}{\omega\mu_o}\hat{z} \times \vec{\nabla}_t\psi = \frac{1}{\eta_\varepsilon}\hat{z} \times \vec{E}_t} \tag{2.44}$$

where $\eta_\varepsilon = \sqrt{\mu_o/\varepsilon} = 377/\sqrt{\varepsilon_r}$ is the intrinsic impedance of the filler dielectric. As in the case of TE and TM waves, so also here we can see that the transverse electric and magnetic fields are at all points mutually perpendicular and $\vec{E}_t$, $\vec{H}_t$, and $\hat{z}$ form a right-handed coordinate system.

From Eq. 2.35,

$$\nabla \cdot \vec{D} = \varepsilon\nabla \cdot \vec{E}_t = -\varepsilon\nabla_t^2\psi = 0 \tag{2.45}$$

From Eq. 2.45 we can see that ψ satisfies the two-dimensional version of the Laplace's equation $\left(\nabla_t^2\psi = 0\right)$ which is the equation governing the electrostatic fields. Once ψ is determined from Eq. 2.45, the electric and magnetic fields can be written from Eqs. 2.43 and 2.44. The fields so obtained are then multiplied by

$e^{j(\omega t - k_\varepsilon z)}$ to represent the time-varying propagating fields rather than the electrostatic fields that Eqs. 2.43-2.45 would otherwise represent. Therefore, the TEM fields can be visualized as the time-varying, $e^{-jk_\varepsilon z}$ propagating fields of the same variation in the cross-sectional plane as the electrostatic fields. It is obvious then that *if TEM modes are to exist in a transmission system, electrostatic fields should be possible in that system. Single conductor systems being electrostatically equipotential are incapable of producing electrostatic field variations in the cross-sectional plane,* hence forbidding the existence of TEM waves in such systems.

The wave impedance Z_{TEM} is defined, as before, by the ratio of the magnitudes of the transverse electric and magnetic fields. From Eqs. 2.43 and 2.44,

$$Z_{TEM} = \frac{|\vec{E}_t|}{|\vec{H}_t|} = \eta_\varepsilon = \frac{377}{\sqrt{\varepsilon_r}} \text{ ohms} \tag{2.46}$$

A more important quantity in TEM modes is the so-called characteristic impedance Z_o of the transmission line. This is the equivalent impedance that will absorb, from the source, the power P that the transmission line is propagating while producing the RF potential $(\psi_2 - \psi_1)$ between its terminals corresponding to the RF potential between the two conductors of the transmission line.

Consequently, the characteristic impedance, Z_o, is given by:

$$Z_o = \frac{(\psi_2 - \psi_1)^2}{2P} \tag{2.47}$$

For multiconductor transmission lines where different potential differences can be established between the various conductors, it is obvious that more than one TEM mode is possible.

2.5 Attenuation in General "Cylindrical" Waveguides

Due to Conductor Losses

The propagation of fields associated with the waveguide mode induces surface currents on the metallic waveguide walls. The currents on the wall surfaces are due to the tangential components of the magnetic field and are given by:

$$\vec{J} = \hat{n} \times \vec{H} \tag{2.48}$$

where $\hat{n}$ is the unit vector normal to the waveguide wall directed towards the