Ex: a) Find $\mathcal{L}\left\{e^{-(t-2)}\int_0^t(\tau-2)\,d\tau\right\}$.

b) Find $v(t)$ if $V(s)=\dfrac{s^2}{(s+1)(s+j)(s-j)}$.

c) Find $\lim\limits_{t\to\infty} v(t)$ if $V(s)=\dfrac{3(s^2+2s+10)}{s^3+3s^2+3s}$.

d) Plot and label the values of the poles and zeros of $V(s)$ in the s plane.

$$V(s)=\frac{s^2+8s+41}{(s^2+1)(s+1)}$$

SOL'N: a) We work from the inside out.

$$\mathcal{L}\{t-2\}=\frac{1}{s^2}-\frac{2}{s}$$

Integrating in time corresponds to multiplying the Laplace transform by $1/s$.

$$\mathcal{L}\left\{\int_0^t(\tau-2)\,d\tau\right\}=\frac{1}{s}\left(\frac{1}{s^2}-\frac{2}{s}\right)=\frac{1}{s^3}-\frac{2}{s^2}$$

Multiplying by a decaying exponential in t corresponds to changing s to $s+a$, and multiplying by e^2 in time just multiplies the Laplace transform by e^2.

$$\mathcal{L}\left\{e^{-(t-2)}\int_0^t(\tau-2)\,d\tau\right\}=e^2\left[\frac{1}{(s+1)^3}-\frac{2}{(s+1)^2}\right]$$

NOTE: The delay identity does not apply. There is no $u(t-2)$ in the problem to delay the turn-on of the signal.

b) We may use a modified partial fraction form that represents the imaginary roots in terms of pure cosine and pure sine.

$$V(s)=\frac{s^2}{(s+1)(s+j)(s-j)}=\frac{s^2}{(s+1)(s^2+1)}=\frac{A}{s+1}+\frac{Bs+C}{s^2+1}$$

To find A, we use the method of multiplying by the pole term and evaluating at the pole.

$$A = V(s)(s+1)\big|_{s=-1} = \left.\frac{s^2}{s^2+1}\right|_{s=-1} = \frac{1}{2}$$

An easy way to find C is to set $s = 0$.

$$V(s=0) = \frac{0^2}{(0+1)(0^2+1)} = 0 = \frac{A}{0+1} + \frac{B(0)+C}{0^2+1} = A + C$$

or

$$C = -A = -\frac{1}{2}$$

To find B, we can set $s = 1$.

$$V(s=0) = \frac{1^2}{(1+1)(1^2+1)} = \frac{1}{4} = \frac{A}{1+1} + \frac{B+C}{1^2+1} = \frac{A+B+C}{2}$$

or

$$V(s=0) = \frac{\frac{1}{2} + B - \frac{1}{2}}{2} = \frac{1}{4}$$

or

$$B = \frac{1}{2}$$

Now we use table entries to complete the inversion, and we multiply by $u(t)$ to emphasize that we cannot say what the value of $f(t)$ is before $t = 0$.

$$\begin{aligned} f(t) &= L^{-1}\left\{\frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{s}{s^2+1} - \frac{1}{2}\frac{1}{s^2+1}\right\} \\ &= \left[\frac{1}{2}e^{-t} + \frac{1}{2}\cos t - \frac{1}{2}\sin t\right]u(t) \end{aligned}$$

c) We use the final value theorem.

$$\lim_{t\to\infty} v(t) = \lim_{s\to 0} sV(s) = \lim_{s\to 0} \frac{s3(s^2+2s+10)}{s^3+3s^2+3s}$$

Now we retain only the lowest power of s in each polynomial.

$$\lim_{t\to\infty} v(t) = \lim_{s\to 0} \frac{s3(10)}{3s}$$

The factors of s top and bottom cancel out.

$$\lim_{t\to\infty} v(t) = \lim_{s\to 0} \frac{3(10)}{3} = 10$$

d) We factor the numerator and denominator. We mark o's on the complex plane for the roots of the numerator (zeros), and x's on the complex plane for the roots of the denominator (poles).

$$V(s) = \frac{s^2 + 8s + 41}{(s^2 + 1)(s + 1)} = \frac{(s + 4)^2 + 5^2}{(s + j)(s - j)(s + 1)} = \frac{(s + 4 + j5)(s + 4 - j5)}{(s + j)(s - j)(s + 1)}$$

The roots are the values of s that make terms equal zero:

zeros: $s = -4 - j5$, $s = -4 + j5$

poles: $s = \mathrm{j}$, $s = -j$, $s = -1$