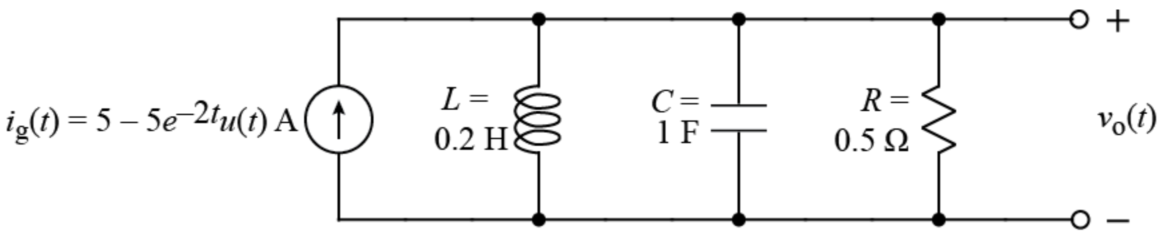
EXAM #3 prob 3 solution

Ex:



Note: The first 5A portion of $i_g(t)$ is always on.

a) Write the Laplace transform $I_g(s)$ of $i_g(t)$.

b) Draw the s-domain equivalent circuit, including source $I_g(s)$, components, initial conditions for L and C, and terminals for $V_o(s)$.

c) Write the Laplace transform $V_o(s)$ of $v_o(t)$. Write your answer as a ratio of polynomials in s with numerical coefficients.

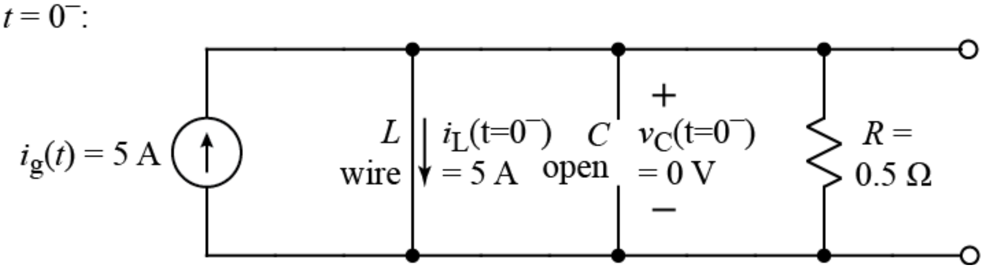
d) Write a numerical time-domain expression for $v_o(t)$ where $t \geq 0$.

SOL'N: a) The 5 A portion of the current source acts like $5u(t)$ for $t = 0^-$ to ∞ where the Laplace transform is computed.

$$I_g(s) = \frac{5}{s} - \frac{5}{s+2}\text{ A}$$

b) To find initial conditions, we use the time-domain circuit at $t = 0^-$ with the C treated as an open circuit and the L treated as a wire. The 5 A portion of the current source will be on, and the second term of the current source will be off.

$t = 0^-$:



The current will all flow in the wire that models the L.

$$i_L(0^-) = 5\text{ A}$$

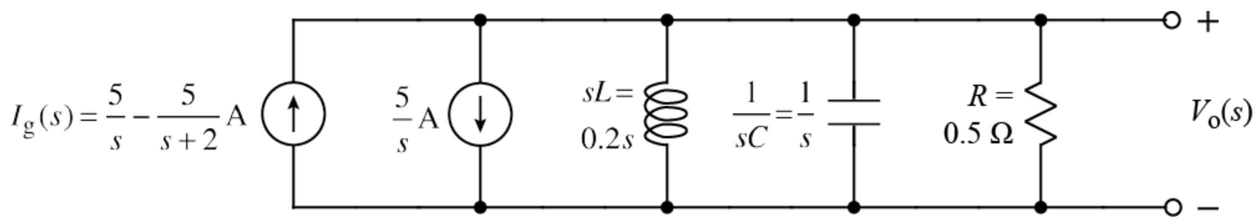
Given the parallel form of the circuit, using a parallel current source for the initial conditions of the L is prudent. The current source corresponds to a step function that turns on a current in parallel with the L at time zero.

$$\frac{i_L(0^-)}{s}$$

The capacitor will have zero initial conditions since it is shorted out by the wire modeling the L.

$$v_C(0^-) = 0\,\text{V}$$

The source for initial conditions on the C may be omitted.



c) We sum the current sources, which results in some convenient cancellation. The output voltage will be the total current times the total parallel impedances.

$$V_o(s) = -\frac{5}{s+2}\text{A}\left(sL \parallel \frac{1}{sC} \parallel R\right) = -\frac{5}{s+2}\left(\frac{1}{\frac{1}{sL}+sC+\frac{1}{R}}\right)$$

We clear the denominator of the denominator and get a coefficient of unity for the s^2 coefficient.

$$V_o(s) = -\frac{5}{s+2}\left(\frac{1}{\frac{1}{sL}+sC+\frac{1}{R}}\right)\frac{s/C}{s/C}$$

or

$$V_o(s) = -\frac{5}{s+2}\left(\frac{s/C}{\frac{1}{LC}+s^2+\frac{1}{RC}s}\right) = -\left(\frac{5}{s+2}\right)\left(\frac{s/C}{s^2+\frac{1}{RC}s+\frac{1}{LC}}\right)$$

or

$$V_o(s) = -\frac{5s}{(s+2)\left(s^2 + \frac{1}{0.5}s + \frac{1}{0.2}\right)} = -\frac{5s}{(s+2)\left(s^2 + 2s + 5\right)}$$

d) We factor the quadratic polynomial in the denominator so it matches the form found in the Laplace transforms of decaying cosine and sine.

$$s^2 + 2s + 5 = (s+1)^2 + 2^2 = (s+a)^2 + \omega^2$$

$$\mathcal{L}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2 + \omega^2}$$

$$\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2 + \omega^2}$$

We expand the output voltage in a modified partial fraction form.

$$V_o(s) = -\frac{5s}{(s+2)\left[(s+1)^2 + 2^2\right]} = \frac{A}{s+2} + \frac{B(s+1) + C(2)}{(s+1)^2 + 2^2}$$

We use the method of multiplying by the pole and evaluating at the pole to find the value of A.

$$A = V_o(s)(s+2)\big|_{s=-2} = -\frac{5s}{\left[(s+1)^2 + 2^2\right]}\Bigg|_{s=-2} = -\frac{5(-2)}{(-2+1)^2 + 2^2}$$

or

$$A = V_o(s)(s+2)\big|_{s=-2} = -\frac{-10}{5} = 2$$

Having found A, we put terms over a common denominator.

$$V_o(s) = \frac{2(s^2 + 2s + 5) + \left[B(s+1) + C(2)\right](s+2)}{(s+2)\left[(s+1)^2 + 2^2\right]}$$

$$= -\frac{5s}{(s+2)\left[(s+1)^2 + 2^2\right]}$$

For the polynomials in the numerator to be equal, the coefficients of each power of s must be equal.

$$2(s^2+2s+5)+\left[B(s+1)+C(2)\right](s+2)=-5s$$

We group the coefficients of powers of s.

$$2(s^2+2s+5)+B(s^2+3s+2)+C(2s+4)=-5s$$

or

$$(2+B)s^2+(4+3B+2C)s+(10+2B+4C)=-5s$$

From the s^2 coefficient, which equals zero (on right side of equation), we find the value of B.

$$B=-2$$

From the constant coefficient, which equals zero (on right side of equation), we find the value of C.

$$10+2(-2)+4C=0$$

or

$$C=-\frac{3}{2}$$

We check our answer by trying some values of s in our original expression for $V_o(s)$ and in our modified partial fraction expression for $V_o(s)$.

$$V_o(s)\Big|_{s=0}=\left[-\frac{5s}{(s+2)\left[(s+1)^2+2^2\right]}\right]\Bigg|_{s=0}=0$$

$$\left[\frac{2}{s+2}+\frac{-2(s+1)-\frac{3}{2}(2)}{s^2+2s+5}\right]\Bigg|_{s=0}=\frac{2}{2}+\frac{-2-3}{5}=0 \text{ works } \surd$$

and

$$V_o(s)\big|_{s=-1} = \left[-\frac{5s}{(s+2)\left[(s+1)^2+2^2\right]} \right]\Bigg|_{s=-1} = \frac{5}{1(4)} = \frac{5}{4}$$

$$\left[\frac{2}{s+2} + \frac{-2(s+1)-\frac{3}{2}(2)}{s^2+2s+5} \right]\Bigg|_{s=-1} = \frac{2}{1} - \frac{3}{1-2+5} = \frac{5}{4} \text{ works } \surd$$

and

$$V_o(s)\big|_{s=-3} = \left[-\frac{5s}{(s+2)\left[(s+1)^2+2^2\right]} \right]\Bigg|_{s=-1} = \frac{-15}{-1(4+4)} = \frac{15}{8}$$

$$\left[\frac{2}{s+2} + \frac{-2(s+1)-\frac{3}{2}(2)}{s^2+2s+5} \right]\Bigg|_{s=-3} = \frac{2}{-1} - \frac{4-3}{9-6+5} = -2 - \frac{1}{8} = \frac{15}{8} \text{ works } \surd$$

Having verified our expansion, we take the inverse Laplace transform.

$$v_o(t) = \mathcal{L}^{-1}\left\{ \frac{2}{s+2} + \frac{-2(s+1)-\frac{3}{2}(2)}{s^2+2s+5} \right\}$$

or

$$v_o(t) = [2e^{-2t} - 2e^{-t}\cos 2t - \frac{3}{2}e^{-t}\sin 2t]u(t)\,\text{V}$$