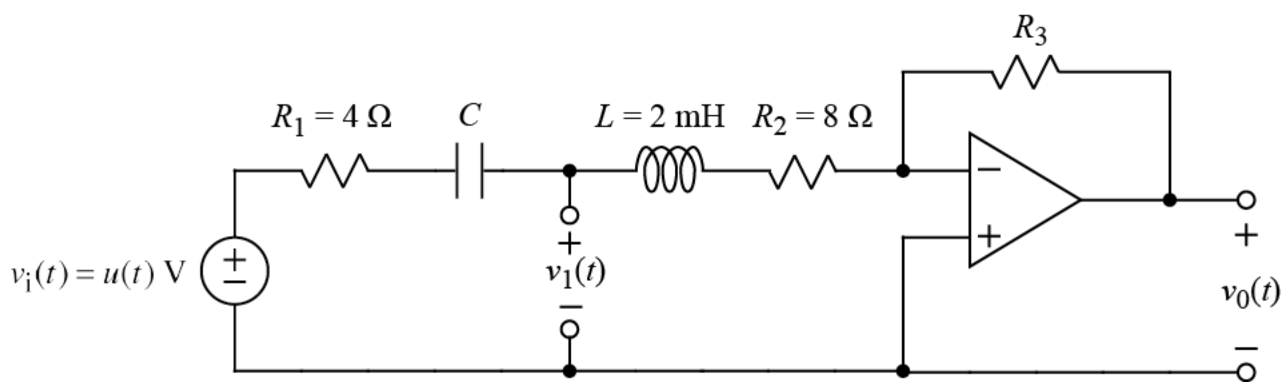
Ex:



The voltage source in the above circuit is off (0 V) for $t < 0$.

An engineer wishes to use the above circuit to create two decaying sinusoidal signals 120° out-of-phase to drive a three-phase motor for a short time. (A third signal that is 120° out-of-phase with the first two may be created by an additional op-amp circuit, not shown, that computes $-v_0 - v_1$.) The signal at $v_0(t)$ will necessarily be a decaying sinusoid of the following form:

$$v_0(t) = -v_m e^{-\alpha t}\sin(\beta t)$$

where v_m, α, and β are positive real-valued constants.

The design problem now is to create a $v_1(t)$ signal that is 120° out-of-phase with $v_0(t)$.

a) Find a symbolic expression for the Laplace-transformed output, $\mathbf{V}_1(s)$, in terms of not more than R_1, R_2, R_3, L, C, and values of sources or constants.

b) Choose a numerical value for C to make

$$v_1(t) = v_m e^{-\alpha t}\cos(\beta t - 30°).$$

Hint: $\cos(A - B) = \cos A \cos B + \sin A \sin B$

bonus) Why could the desired $v_1(t)$ not be obtained if the positions of the L and C were reversed?

SOL'N: a) The input voltage source is a step function that Laplace transforms to $1/s$.

$$\mathbf{V}_i(s) = \frac{1}{s}$$

Before time zero, the input voltage is zero and it follows that initial conditions for both the L and C are zero.

At the – input of the op-amp, we have the same voltage (because of the negative feedback) as at the + input, namely zero volts.

We can express the current flowing toward the – input as the input voltage divided by the sum of impedances up to the – input. This is true in the Laplace domain and is just an example of Ohm's law.

$$\mathbf{I}(s) = \frac{1}{s}\frac{1}{sL + R + \dfrac{1}{sC}} = \frac{1/L}{s^2 + \dfrac{R}{L}s + \dfrac{1}{LC}}$$

where R represents $R_1 + R_2$.

To find $\mathbf{V}_1(s)$, we observe that we may use the voltage drop across L and R_2. Again, we use Ohm's law, multiplying the impedances of L and R_2 by $\mathbf{I}(s)$.

$$\mathbf{V}_1(s) = \mathbf{I}(s)(sL + R_2) = \frac{s + \dfrac{R_2}{L}}{s^2 + \dfrac{R}{L}s + \dfrac{1}{LC}} = \frac{s + \dfrac{R_2}{L}}{\left(s + \dfrac{R}{2L}\right)^2 + \dfrac{1}{LC} - \left(\dfrac{R}{2L}\right)^2}$$

The second form for the answer will figure into our solution to (b).

b) Using the hint, we rewrite the expression for v_1 in terms of sine and cosine.

$$v_1(t) = v_m e^{-\alpha t}[\cos(\beta t)\cos(30°) + \sin(\beta t)\sin(30°)]$$

or

$$v_1(t) = v_m e^{-\alpha t}\left[\cos(\beta t)\frac{\sqrt{3}}{2} + \sin(\beta t)\frac{1}{2}\right]$$

We Laplace transform the expression for $v_1(t)$.

$$\mathbf{V}_1(s) = v_m\left[\frac{\sqrt{3}}{2}\frac{s+\alpha}{(s+\alpha)^2 + \beta^2} + \frac{1}{2}\frac{\beta}{(s+\alpha)^2 + \beta^2}\right]$$

Matching the denominator to our answer from (a), we identify the values of α and β.

$$\alpha = \frac{R}{2L}$$

$$\beta^2 = \frac{1}{LC} - \left(\frac{R}{2L}\right)^2$$

We can calculate the numerical value of α.

$$\frac{R}{2L} = \frac{4+8}{2(2\text{m})}\text{r/s} = 3\text{k}$$

Now we turn our attention to the numerator of $\mathbf{V}_1(s)$.

$$\mathbf{V}_1(s) = v_m \frac{1}{2}\left[\frac{\sqrt{3}(s+\alpha)+\beta}{(s+\alpha)^2+\beta^2}\right] = \left[\frac{v_m \frac{\sqrt{3}}{2} s + v_m \frac{\sqrt{3}}{2}\alpha + v_m \frac{1}{2}\beta}{(s+\alpha)^2+\beta^2}\right]$$

From the solution to (a), the coefficient of s is unity, which dictates the necessary value of v_m.

$$v_m = \frac{2}{\sqrt{3}}$$

Now we consider the constant term of the numerator, which must map the solution from (a). Using our value of v_m and the solution to (a) gives the following equation.

$$\alpha + \frac{1}{\sqrt{3}}\beta = \frac{R_2}{L}$$

or

$$\frac{R_1+R_2}{2L} + \frac{1}{\sqrt{3}}\beta = \frac{R_2}{L}$$

or, if we subtract $R_2/2L$ from both sides, we have the following equation:

$$\frac{R_1}{2L} + \frac{1}{\sqrt{3}}\beta = \frac{R_2}{2L}$$

A few calculations:

$$\frac{R_1}{2L} = \frac{4}{2(2\text{m})}\text{r/s} = 1\text{k} \quad \text{and} \quad \frac{R_2}{2L} = \frac{8}{2(2\text{m})}\text{r/s} = 2\text{k}$$

Using these values, we have an equation for β.

$$1\text{k}+\frac{1}{\sqrt{3}}\beta=2\,\text{k}$$

or

$$\frac{1}{\sqrt{3}}\beta=1\text{k}$$

or

$$\frac{1}{3}\beta^2=1\text{M}$$

or, using the expression for β from earlier, we have the following:

$$\beta^2=\frac{1}{LC}-\left(\frac{R}{2L}\right)^2=3\text{M}$$

or

$$\frac{1}{LC}-(3\text{k})^2=3\text{M}$$

or

$$\frac{1}{2\text{m}C}=12\text{M}$$

Finally, we can solve for C.

$$C=\frac{1}{2\text{m}12\text{M}}=\frac{1}{24\text{k}}\approx 41.7\mu\text{F}$$

bonus) With the C on the right, $v_1(t)$ would end up at 1 V as the C would charge. Thus, the signal could not be a decaying sinusoid. It would have a DC offset.