Ex: Show that the following identity is valid:

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$$

SOL'N: This identity is easier to prove if we start we the right side and show that it is equal to the left side.

$$-\frac{d}{ds}F(s) = \frac{-d}{ds}\mathcal{L}\{f(t)\} = \frac{-d}{ds}\int_{0^-}^{\infty} f(t)e^{-st}dt$$

Because differentiation and integration are linear operators, and because s is not a function of t, we can exchange the order of differentiation and integration.

$$\frac{-d}{ds}\int_{0^-}^{\infty} f(t)e^{-st}dt = \int_{0^-}^{\infty}\frac{-d}{ds}\left[f(t)e^{-st}\right]dt$$

$f(t)$ acts like a constant with respect to differentiation by s, and the exponential has a simple derivative.

$$\int_{0^-}^{\infty}\frac{-d}{ds}\left[f(t)e^{-st}\right]dt = \int_{0^-}^{\infty} -f(t)(-t)e^{-st}dt$$

The minus signs cancel, and we complete the proof.

$$\int_{0^-}^{\infty} -f(t)(-t)e^{-st}dt = \int_{0^-}^{\infty} tf(t)e^{-st}dt = \mathcal{L}\{tf(t)\}$$