EX: Find the inverse Laplace transform for the following expression:

$$F(s) = -\frac{3s^2+3}{s^4}$$

SOL'N: When there is a repeated root, each power of the root appears in the partial fraction expansion, (along with the usual partial fraction terms for any other distinct roots, if any—in this case there are no others).

$$F(s) = \frac{A_1}{s^4} + \frac{A_2}{s^3} + \frac{A_3}{s^2} + \frac{A_4}{s}$$

We find A_1 by multiplying by the highest power of the root and evaluating at the value of the root:

$$A_1 = s^4 F(s)\big|_{s=0} = -(3s^2+3)\Big|_{s=0} = -3$$

To see why this works, we can perform the same operation on the partial fraction expression:

$$s^4 F(s)\big|_{s=0} = \left(\frac{s^4 A_1}{s^4} + \frac{s^4 A_2}{s^3} + \frac{s^4 A_3}{s^2} + \frac{s^4 A_4}{s}\right)\Bigg|_{s=0}$$

or

$$s^4 F(s)\big|_{s=0} = A_1 + 0\cdot A_2 + 0\cdot A_3 + 0\cdot A_4 = A_1$$

We might suppose that we would find A_2 by multiplying by s^3 and evaluating at the root, but this causes the A_1 term to be divided by zero. Thus, this approach is *incorrect*.

Instead, we differentiate $s^4F(s)$ and evaluate at the value of the root. Symbolically, we have the following equation:

$$A_2 = \left\{\frac{d}{dt}\left[s^4 F(s)\right]\right\}\Bigg|_{s=0} = \left[\frac{d}{dt}\left(\frac{s^4 A_1}{s^4} + \frac{s^4 A_2}{s^3} + \frac{s^4 A_3}{s^2} + \frac{s^4 A_4}{s}\right)\right]_{s=0}$$

or

$$A_2 = \left\{\frac{d}{dt}\left[s^4 F(s)\right]\right\}\Bigg|_{s=0} = \left[\frac{d}{dt}\left(A_1 + sA_2 + s^2 A_3 + s^3 A_4\right)\right]_{s=0}$$

or

$$A_2 = \left[0 + A_2 + 2sA_3 + 3s^2 A_4\right]\Big|_{s=0} = 0 + A_2 + 2\cdot 0\cdot A_3 + 3\cdot 0^2 A_4 = A_2$$

NOTE: We differentiate first. Then we substitute the value of the root. If we substitute the value of the root first, the derivative will always be zero, (since the derivative of a constant is zero).

Here, we obtain the following result:

$$A_2 = \left\{\frac{d}{dt}\left[s^4 F(s)\right]\right\}\Big|_{s=0} = \left\{\frac{d}{dt}\left[-(3s^2+3)\right]\right\}\Big|_{s=0} = -6s\big|_{s=0} = 0$$

To find A_3, we differentiate $s^4F(s)$ again and evaluate at the value of the root, but now we must divide by two:

$$A_3 = \frac{1}{2}\left\{\frac{d^2}{dt^2}\left[s^4 F(s)\right]\right\}\Big|_{s=0} = \frac{1}{2}\left[\frac{d}{dt}\left(A_2 + 2sA_3 + 3s^2 A_4\right)\right]_{s=0}$$

or

$$A_3 = \frac{1}{2}\{2A_3 + 2\cdot 3sA_4\}\big|_{s=0} = \frac{1}{2}\{2A_3 + 2\cdot 3\cdot 0\cdot A_4\}\big|_{s=0} = A_3$$

Here, we obtain the following result:

$$A_3 = \frac{1}{2}\left\{\frac{d^2}{dt^2}\left[s^4 F(s)\right]\right\}\Big|_{s=0} = \frac{1}{2}\left\{\frac{d}{dt}[-6s]\right\}\Big|_{s=0} = \frac{1}{2}\{-6\}\big|_{s=0} = -3$$

To find A_4, we differentiate $s^4F(s)$ again and evaluate at the value of the root, but we must divide by three factorial. (In general, we divide by $n!$):

$$A_4 = \frac{1}{3!}\left\{\frac{d^3}{dt^3}\left[s^4 F(s)\right]\right\}\Big|_{s=0} = \frac{1}{3!}\left[\frac{d}{dt}(2A_3 + 2\cdot 3sA_4)\right]_{s=0}$$

or

$$A_4 = \frac{1}{3!}[2\cdot 3A_4]\big|_{s=0} = A_4$$

Here, we obtain the following result:

$$A_4 = \frac{1}{3!}\left\{\frac{d^3}{dt^3}\left[s^4 F(s)\right]\right\}\Bigg|_{s=0} = \frac{1}{3!}\left\{\frac{d}{dt}[-6]\right\}\Bigg|_{s=0} = 0$$

We have completed the partial fraction expansion:

$$F(s) = \frac{-3}{s^4} + \frac{0}{s^3} + \frac{-3}{s^2} + \frac{0}{s}$$

NOTE: This problem is simple enough that we could obtain the same result by rewriting $F(s)$, (but this is usually not the case):

$$F(s) = -\frac{3s^2+3}{s^4} = \frac{-3}{s^2} + \frac{-3}{s^4}$$

NOTE: Other methods of finding partial fraction coefficients, such as substituting specific values of s (not equal to roots), will also lead to the same results. In a problem with four coefficients, however, we would obtain four equations in four unknowns. This may prove more cumbersome than the above approach.

For the inverse transform, we use the following transform pair:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^n}\right\} = \frac{t^{n-1}}{(n-1)!}e^{-at}$$

This yields our final result:

$$f(t) = -3\frac{t^3}{3!} - 3t = -\frac{t^3}{2} - 3t$$