HOMEWORK #12 prob 3d solution

EX: Find the inverse Laplace transform for the following expression:

$$F(s) = \frac{6s^2 + 36s + 198}{(s+3)(s^2 + 6s + 45)}$$

SOL'N: One approach to this problem is to use a partial fraction expansion with a coefficient for every root. Because the constant term of the quadratic in the denominator is larger than the square of half the middle coefficient, the quadratic has complex roots.

$$s^2 + 6s + 45 = (s + 3 + j6)(s + 3 - j6) = (s + a + j\omega)(s + a - j\omega)$$

The coefficients for the complex roots will be complex conjugates.

$$F(s) = \frac{6s^2 + 36s + 198}{(s+3)(s^2 + 6s + 45)} = \frac{A}{s+3} + \frac{B}{s+3+j6} + \frac{B^*}{s+3-j6}$$

We multiply by a root term and then evaluate at the root value. For the first root, we have the following calculation:

$$A = (s+3)F(s)\Big|_{s=-3} = \left.\frac{6s^2 + 36s + 198}{s^2 + 6s + 45}\right|_{s=-3}$$

or

$$A = \frac{6(-3)^2 + 36(-3) + 198}{(-3)^2 + 6(-3) + 45} = \frac{54 - 108 + 198}{9 - 18 + 45} = \frac{144}{36} = 4$$

The inverse transform for this term is an exponential decay:

$$\mathcal{L}^{-1}\left[\frac{4}{s+3}\right] = 4e^{-3t}u(t)$$

For the second root, we have the following calculation:

$$B = (s + 3 + j6)F(s)\Big|_{s=-3-j6} = \left.\frac{6s^2 + 36s + 198}{(s+3)(s+3-j6)}\right|_{s=-3-j6}$$

or

$$B = \frac{6(-3-j6)^2 + 36(-3-j6) + 198}{(-3-j6+3)(-3-j6+3-j6)} = \frac{6\left[(-3-j6)^2 + 6(-3-j6) + 33\right]}{-j6(-j12)}$$

We can cancel a factor of 6 on top and bottom:

$$B = \frac{(-3-j6)^2 + 6(-3-j6) + 33}{-12} = \frac{9 + j36 - 36 + -18 - j36 + 33}{-12}$$

or

$$B = \frac{-12}{-12} = 1$$

Since this coefficient is real, its complex conjugate has the same value:

$$B^* = 1$$

Using the complex root identity for decaying cos() and sin(), the inverse transform for the complex root terms are as follows:

$$\mathcal{L}^{-1}\left[\frac{1}{s+3+j6} + \frac{1}{s+3-j6}\right] = 2e^{-3t}\cos(6t)u(t)$$

Combining the above results, we have our final answer:

$$\mathcal{L}^{-1}[F(s)] = \left[4e^{-3t} + 2e^{-3t}\cos(6t)\right]u(t)$$

Another approach to this problem is to replace $s + 3$ with s or, equivalently, replace s with $s - 3$ and multiply in the time domain by e^{-3t} in the final step. In other words, this approach exploits the "multiply by e^{-at}" identity:

$$\mathcal{L}\left[e^{-at}v(t)\right] = V(s)\Big|_{s+a \text{ replaces } s}$$

So we replace $s + 3$ with s:

$$F_2(s) = \frac{6(s-3)^2 + 36(s-3) + 198}{s(s^2+36)}$$

We can factor out a 6 from the top to make the numbers smaller in our calculations:

$$F_2(s) = \frac{6\left[(s-3)^2 + 6(s-3) + 33\right]}{s(s^2+36)}$$

Now we work on this expression and match the complex root terms to a cosine and sine:

$$F_2(s) = \frac{6\left[(s-3)^2 + 6(s-3) + 33\right]}{s(s^2+36)} = \frac{A}{s} + \frac{Bs}{s^2+6^2} + \frac{C\omega}{s^2+6^2}$$

Recall the transform pairs for cosine and sine are as follows:

$$\mathcal{L}[\cos(\omega t)] \ = \ \frac{s}{s^2+6^2} \quad \text{and} \quad \mathcal{L}[\sin(\omega t)] \ = \ \frac{\omega}{s^2+6^2}$$

We can find A by the usual method of multiply by the root term and evaluating at the root value:

$$A = sF_2(s)\big|_{s=0} = \left.\frac{6\left[(s-3)^2 + 6(s-3) + 33\right]}{(s^2+36)}\right|_{s=0} = \frac{6\left[(-3)^2 + 6(-3) + 33\right]}{36}$$

or

$$A = \frac{6[9-18+33]}{36} = \frac{6 \cdot 24}{36} = 4$$

To find B and C, we can use a common denominator.

$$F_2(s) = \frac{6\left[(s-3)^2 + 6(s-3) + 33\right]}{s\left(s^2+6^2\right)} = \frac{A\left(s^2+6^2\right) + Bs^2 + C\omega s}{s\left(s^2+6^2\right)}$$

Starting with the highest power of s in the numerator, we match coefficients of each power of s:

$$6s^2 = As^2 + Bs^2 = 4s^2 + Bs^2$$

The value of B is easily computed:

$$B = 2$$

We find the value of C by matching the coefficient of s in the numerator:

$$6[-6s + 6s] = C\omega s$$

We conclude that C is zero:

$$C = 0$$

Now we have the partial fraction expansion for $F_2(s)$:

$$F_2(s) = \frac{4}{s} + \frac{2s}{s^2 + 6^2}$$

Taking the inverse transform and multiplying by e^{-3t}, we have our final answer, which is the same as before:

$$\mathcal{L}^{-1}\left[F(s)\right] = \left[4e^{-3t} + 2e^{-3t}\cos(6t)\right]u(t)$$

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