Using Frequency-domain (Bode Plot) Design for the Double Integrator

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Double Integrator

A very common system and a difficult design problem.

It's Newton's fault: $F = m \cdot a = m \cdot \frac{d^2}{dt^2} x$ $\qquad x = \frac{1}{m} \cdot \left(\int \int F \, dt \, dt \right)$ $\qquad \mathbf{X}(s) = \mathbf{F}(s) \cdot \frac{1}{m \cdot s^2}$

Same for angular motion: $T = J \cdot \alpha = J \cdot \frac{d^2}{dt^2} \theta$ $\qquad$ & $\mathbf{P}(s) = \frac{1}{m \cdot s^2}$

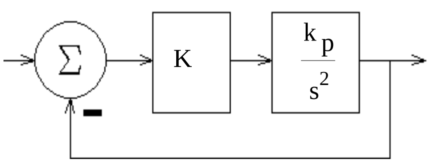
This problem arises anytime force is the input and position is the output.

Force is the ONLY way to get the motion of any object to change, so yes, this is a common problem.

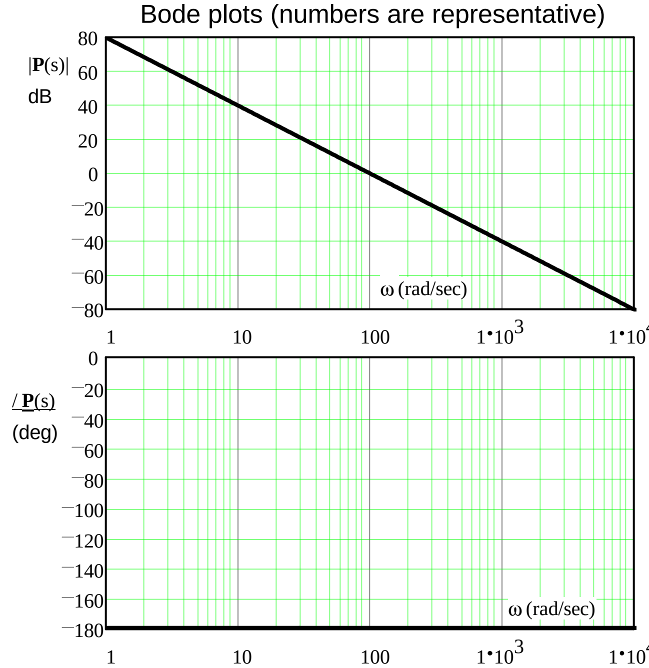
In the Inverted Pendulum lab, the movement of the base is simplified to a first-order system to avoid the difficulties that come from this very issue.

The example used in section 5.3.9 is a VERY REAL example.

in general: $\mathbf{P}(s) = \frac{k_p}{s^2}$

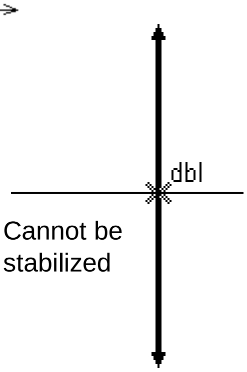


Bode plots (numbers are representative)



Root-locus for the double-integrator in a closed-loop (CL) feedback system.

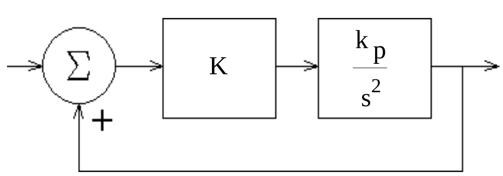
Cannot be stabilized



$$\mathbf{H}(s) = \frac{K \cdot \frac{k_p}{s^2}}{1 + K \cdot \frac{k_p}{s^2}} \cdot \left(\frac{s^2}{s^2} \right) = \frac{K \cdot k_p}{s^2 + K \cdot k_p}$$

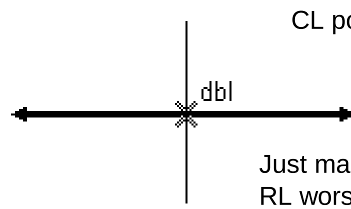
MUST use a compensator.

If the angle is always 180, then wouldn't positive feedback work?



$$\mathbf{H}(s) = \frac{K \cdot \frac{k_p}{s^2}}{1 - K \cdot \frac{k_p}{s^2}} \cdot \left(\frac{s^2}{s^2} \right) = \frac{K \cdot k_p}{s^2 - K \cdot k_p}$$

CL poles $\pm \sqrt{K \cdot k_p}$



Just makes the RL worse.

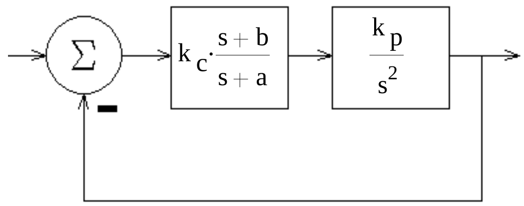
Positive feedback is similar to negative gain, which makes root-locus rules work backwards, here the real-axis rule:

Given the issues with a PD (the differentiator). lets use a Lead controller.

Lead controller

See section 5.3.9

$$C(s) = k_c \cdot \frac{s+b}{s+a}$$



Put the two together,

$$G(s) = k_c \cdot \frac{s+b}{s+a} \cdot \frac{k_p}{s^2} = k_p \cdot k_c \cdot \frac{s+b}{s^2 \cdot (s+a)}$$

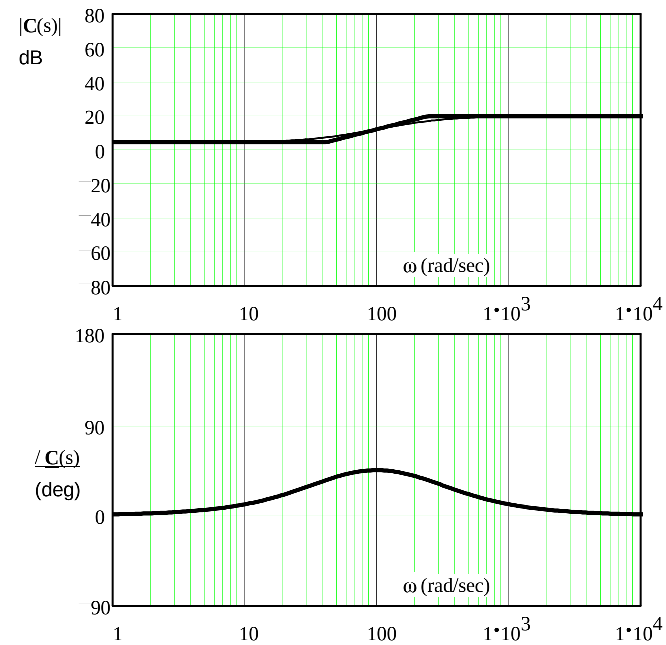
But now the maximum phase angle difference from 180 doesn't occur where the magnitude crosses 0dB.

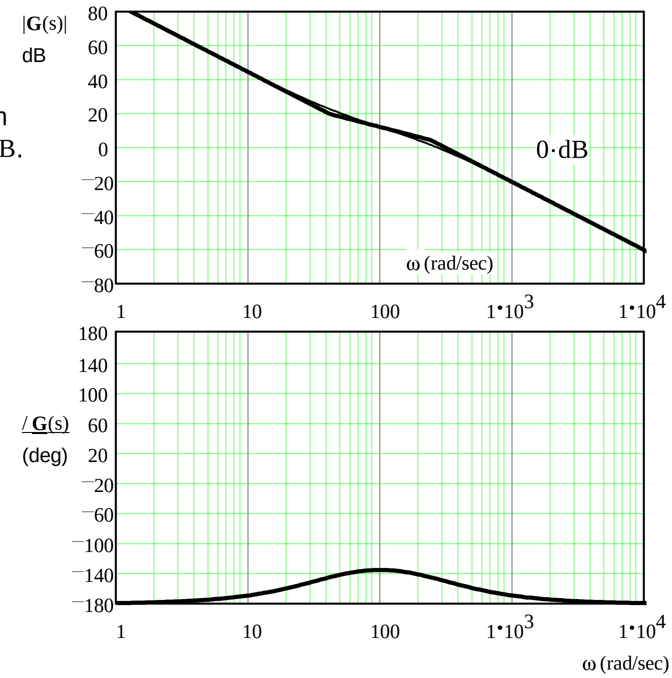
This problem is resolved in the math shown in the book, which makes:

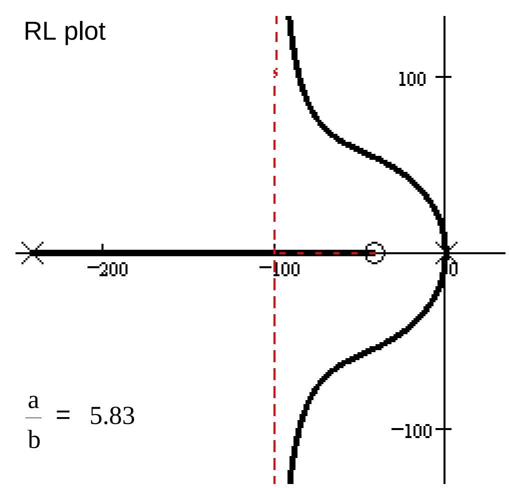
$$\omega_c = \omega_p$$

freq. of maximum phase difference = freq. where **G**(s) crosses 0dB.

Bode plots (numbers are representative)







$$\frac{a}{b} = 5.83$$

The Bottom Line

I've combined information from the table in section 5.3.7 with the table in section 5.3.9.

For double integrator problem; ζ: approximation from simpler system of section 5.3.7

$\left(\frac{a}{b}\right)$	ϕ_p = PM	ζ	%OS = PO
5.83	45^o	0.44	$20.5\cdot\%$
9	53.1^o	0.55	$14\cdot\%$
13.9	60^o	0.6	$9.5\cdot\%$

= percent overshoot based on ζ approx.

PM, ζ relationship is also shown in section 5.3.7, 2nd eq. (5.63)

1. Select your a/b ratio, use this ratio as a single number in following equations.

   use $\left(\frac{a}{b}\right)$ as a single number

   Or use eq. 5.73

   Extension of table using approximate relationship between PM and overshoot developed in section 5.3.7

2. Use eq. 5.75 to relate ω_c to k_p and k_c. $\frac{k_p \cdot k_c}{\omega_c^2}\cdot\sqrt{\frac{b}{a}} = 1$ OR, rearranged: $\omega_p = \omega_c = \sqrt{k_p \cdot k_c \cdot \sqrt{\frac{b}{a}}}$

   Note: $\frac{b}{a} = \frac{1}{\left(\frac{a}{b}\right)}$

   Depending on your knowns and unknowns, other rearrangements may be useful:

   $$k_p \cdot k_c = \omega_c^2 \cdot \sqrt{\frac{a}{b}} \qquad k_p = \frac{\omega_c^2}{k_c}\cdot\sqrt{\frac{a}{b}} \qquad k_c = \frac{\omega_c^2}{k_p}\cdot\sqrt{\frac{a}{b}}$$

   To get some answers, I arbitrarily used: $\omega_c := 10$ $\quad k_p := 1$ and found k_c from the eq. above

3. Find: $a = \omega_c \cdot \sqrt{\frac{a}{b}} = \omega_p \cdot \sqrt{\frac{a}{b}}$ $\qquad b = \omega_c \cdot \sqrt{\frac{b}{a}} = \omega_p \cdot \sqrt{\frac{b}{a}}$

   the pole location of **C**(s) $\qquad$ the zero location of **C**(s)

Problem 5.14 in the text shows that the approximations of overshoot given in the table above are not very good (off by about a factor of 2), but, those predicted by the second-order approximation are even worse (b/c of zero close to origin).

Why Bode Plots?

1. Provides a method to find the approximate transfer function as used in the Flexible Beam lab.
2. Terms GM and PM are in wide use and you need to know what they mean.
3. Sometimes used for design method as in the Flexible Beam lab.

Example Problem 5.14 in the text.

a) Consider the lead controller for the double integrator. For the design that makes the crossover frequency equal to ω_C, obtain the polynomial that specifies the closed-loop poles (as a function of a/b and ω_C). Show that one closed-loop pole is at s = - ω_C no matter what a/b is.

$$\mathbf{G_c}(s) = \mathbf{P}(s)\cdot\mathbf{C}(s) = \frac{k_p}{s^2}\cdot k_c\cdot\frac{s+b}{s+a}$$

Denominator of the closed-loop transfer function: $\mathbf{D_G}+\mathbf{N_G} = s^2\cdot(s+a)+k_p\cdot k_c\cdot(s+b)$

$= s^3+a\cdot s^2+k_p\cdot k_c\cdot(s+b) \quad = 0$ to find poles

Substitute: $a = \omega_c\cdot\sqrt{\frac{a}{b}}$ $\quad b = \omega_c\cdot\sqrt{\frac{b}{a}}$ $\quad k_c = \frac{\omega_c^2}{k_p}\cdot\sqrt{\frac{a}{b}}$ eq. 5.70 in book.

$$0 = s^3+\omega_c\cdot\sqrt{\frac{a}{b}}\cdot s^2+k_p\cdot\left(\frac{\omega_c^2}{k_p}\cdot\sqrt{\frac{a}{b}}\right)\cdot\left(s+\omega_c\cdot\sqrt{\frac{b}{a}}\right) = s^3+\omega_c\cdot\sqrt{\frac{a}{b}}\cdot s^2+\omega_c^2\cdot\sqrt{\frac{a}{b}}\cdot s+\omega_c^2\cdot\sqrt{\frac{a}{b}}\cdot\omega_c\cdot\sqrt{\frac{b}{a}}$$

$$= s^3+\omega_c\cdot\sqrt{\frac{a}{b}}\cdot s^2+\omega_c^2\cdot\sqrt{\frac{a}{b}}\cdot s+\omega_c^3$$

Polynomial division: divisor $s+\omega_c$

Quotient: $s^2+\omega_c\cdot\left(\sqrt{\frac{a}{b}}-1\right)\cdot s+\omega_c^2$

$$s^3+\omega_c\cdot\sqrt{\frac{a}{b}}\cdot s^2+\omega_c^2\cdot\sqrt{\frac{a}{b}}\cdot s+\omega_c^3$$

$$-s^3-\omega_c\cdot s^2$$

$$\left(\omega_c\cdot\sqrt{\frac{a}{b}}-\omega_c\right)\cdot s^2+\omega_c^2\cdot\sqrt{\frac{a}{b}}\cdot s+\omega_c^3$$

$$-\omega_c\cdot\left(\sqrt{\frac{a}{b}}-1\right)\cdot s^2-\omega_c^2\cdot\left(\sqrt{\frac{a}{b}}-1\right)\cdot s$$

$$\omega_c^2\cdot s+\omega_c^3$$

$$\omega_c^2\cdot s+\omega_c^3$$

$$0$$

No remainder, QED

b) Compute the other closed-loop poles, as functions of ω_C, when a/b = 5.83, 9, and 13.9.

The "other" roots are the roots of the quotient. $\quad 0 = s^2+\omega_c\cdot\left(\sqrt{\frac{a}{b}}-1\right)\cdot s+\omega_c^2$

a/b = 5.83 $\quad \left(\sqrt{\frac{a}{b}}-1\right) = \left(\sqrt{5.83}-1\right) = 1.415$

$$s = \left[\frac{-\omega_c\cdot\left(\sqrt{5.83}-1\right)}{2}+\frac{1}{2}\cdot\sqrt{\left[\omega_c\cdot\left(\sqrt{5.83}-1\right)\right]^2-4\cdot\omega_c^2}\right] = \frac{-\omega_c\cdot\left(\sqrt{5.83}-1\right)}{2}+\frac{1}{2}\cdot\omega_c\cdot\sqrt{\left(\sqrt{5.83}-1\right)^2-4}$$

$$= \frac{-\left(\sqrt{5.83}-1\right)}{2}+\frac{1}{2}\cdot\sqrt{\left(\sqrt{5.83}-1\right)^2-4} = -0.707+0.707j \qquad (-0.7071-0.7071\cdot j)\cdot\omega_c \quad \& \quad (-0.7071+0.7071\cdot j)\cdot\omega_c$$

a/b = 9 $\quad \frac{-\left(\sqrt{9}-1\right)}{2} = -1 \qquad \frac{\sqrt{\left(\sqrt{9}-1\right)^2-4}}{2} = 0 \qquad -\omega_c \quad \& \quad -\omega_c$

a/b = 13.9 $\quad \frac{-\left(\sqrt{13.9}-1\right)}{2}+\frac{1}{2}\cdot\sqrt{\left(\sqrt{13.9}-1\right)^2-4} = -0.436 \qquad \frac{-\left(\sqrt{13.9}-1\right)}{2}-\frac{1}{2}\cdot\sqrt{\left(\sqrt{13.9}-1\right)^2-4} = -2.292$

$-0.436\cdot\omega_c \quad \& \quad -2.292\cdot\omega_c$

For plots: $t := 0.01, 0.02 .. 1.5$

c) Use Matlab SISO or other software of your choice to confirm the results of part c) and the % overshoot figures expected from the phase margins by the second-order approximation. ($20\cdot\%$ $14\cdot\%$ $9.5\cdot\%$)

$$\mathbf{P}(s)\cdot\mathbf{C}(s) = \frac{k_p}{s^2}\cdot k_c\cdot\frac{s+b}{s+a} \qquad H(s) = \frac{P(s)\cdot C(s)}{1+P(s)\cdot C(s)} = \frac{k_p\cdot k_c\cdot(s+b)}{(s+a)\cdot s^2 + k_p\cdot k_c\cdot(s+b)}$$

$\mathbf{X}(s) = \frac{1}{s}$ the unit step function

$$\mathbf{Y}(s) = \mathbf{X}(s)\cdot\mathbf{H}(s) = \frac{1}{s}\cdot\frac{k_p\cdot k_c\cdot(s+b)}{(s+a)\cdot s^2 + k_p\cdot k_c\cdot(s+b)} = \frac{1}{s}\cdot\frac{k_p\cdot k_c\cdot(s+b)}{(s+a)\cdot s^2 + k_p\cdot k_c\cdot(s+b)}$$

$$= \frac{1}{s}\cdot\frac{k_p\cdot k_c\cdot(s+b)}{(s+\omega_c)\cdot\left[s^2+\omega_c\cdot\left(\sqrt{\frac{a}{b}}-1\right)\cdot s+\omega_c^2\right]}$$

I will set: $\omega_c := 10$ $\quad k_p := 1$

First case: $\frac{a}{b} = a\_b := 5.83 \qquad a = \omega_c\cdot\sqrt{\frac{a}{b}} = a := \omega_c\cdot\sqrt{a\_b} \qquad a = 24.145$

$$b = \frac{\omega_c}{\sqrt{\frac{a}{b}}} = b := \frac{\omega_c}{\sqrt{a\_b}} \qquad b = 4.142$$

$$k_c = \frac{\omega_c^2}{k_p}\cdot\sqrt{\frac{a}{b}} = k_c := \frac{\omega_c^2}{k_p}\cdot\sqrt{a\_b} \qquad k_c = 241.454$$

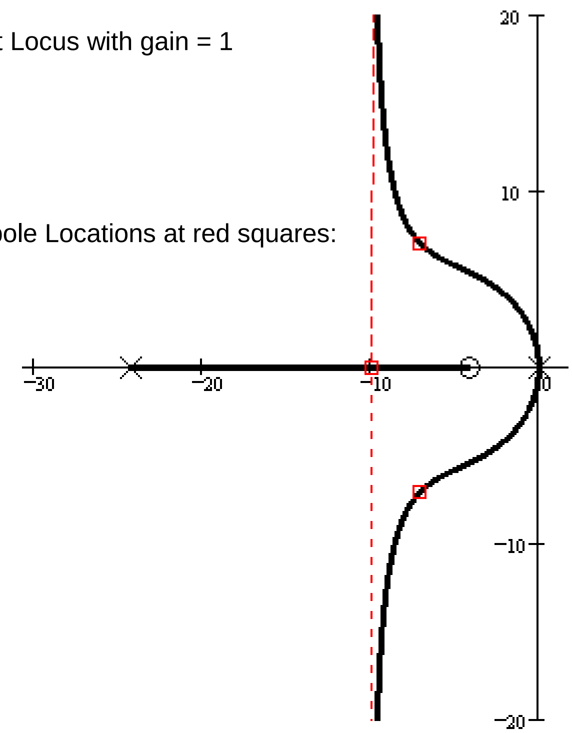
$$\mathbf{G_c}(s) = \frac{k_p}{s^2}\cdot k_c\cdot\frac{s+b}{s+a} = \frac{241.454\cdot(s+4.142)}{s^2\cdot(s+24.145)}$$

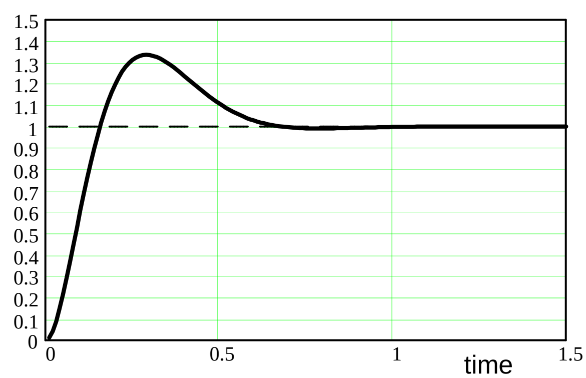
Expected overshoot

$$\zeta := \frac{45\cdot deg}{100\cdot deg} \qquad \zeta = 0.45 \qquad 100\cdot\%\cdot e^{\frac{-\zeta\cdot\pi}{\sqrt{1-\zeta^2}}} = 20.535\cdot\%$$

Root Locus with gain = 1

CL pole Locations at red squares:





% overshoot ≃ $35\cdot\%$

Second case:

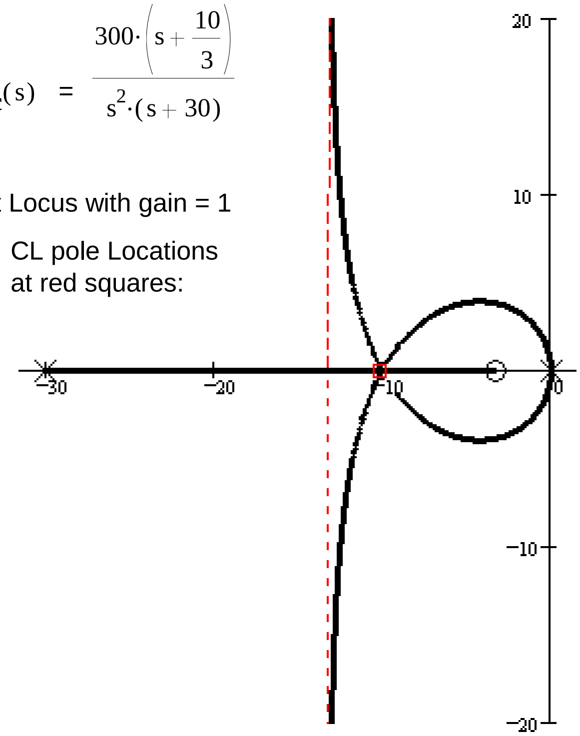
$\frac{a}{b} = a\_b := 9$ $\quad a := \omega_c \cdot \sqrt{a\_b}$ $\quad b := \frac{\omega_c}{\sqrt{a\_b}}$ $\quad b = 3.333$ $\quad k_c := \frac{\omega_c^2}{k_p} \cdot \sqrt{a\_b}$ $\quad k_c = 300$

$a = 30$

$$G_c(s) = \frac{300 \cdot \left(s + \frac{10}{3}\right)}{s^2 \cdot (s + 30)}$$

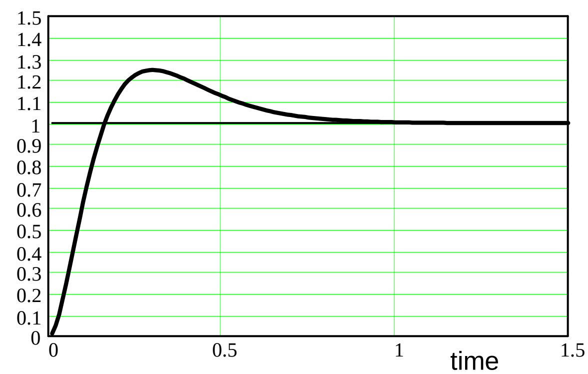
Root Locus with gain = 1

CL pole Locations at red squares:



Expected overshoot

$\zeta := \frac{53.1 \cdot deg}{100 \cdot deg}$ $\quad \zeta = 0.531$ $\quad 100 \cdot \% \cdot e^{\frac{-\zeta \cdot \pi}{\sqrt{1-\zeta^2}}} = 13.964 \cdot \%$



% overshoot $\simeq$ 27·%

Third case:

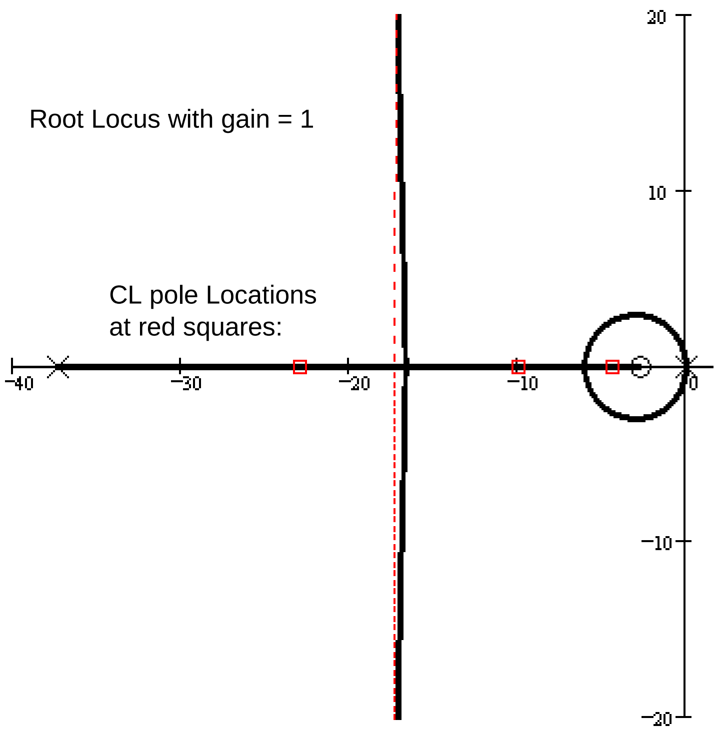
$\frac{a}{b} = a\_b := 13.9$ $\quad a := \omega_c \cdot \sqrt{a\_b}$ $\quad b := \frac{\omega_c}{\sqrt{a\_b}}$ $\quad b = 2.682$ $\quad k_c := \frac{\omega_c^2}{k_p} \cdot \sqrt{a\_b}$ $\quad k_c = 372.827$

$a = 37.283$

$$G_c(s) = \frac{372.827 \cdot (s + 2.682)}{s^2 \cdot (s + 37.283)}$$

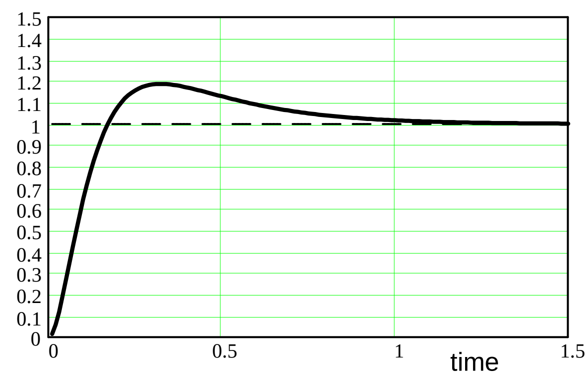
Root Locus with gain = 1

CL pole Locations at red squares:



Expected overshoot

$\zeta := \frac{60 \cdot deg}{100 \cdot deg}$ $\quad \zeta = 0.6$ $\quad 100 \cdot \% \cdot e^{\frac{-\zeta \cdot \pi}{\sqrt{1-\zeta^2}}} = 9.478 \cdot \%$



% overshoot $\simeq$ 21·%

Actual overshoots are much larger than expected by the table above, but, overshoots predicted by the second-order approximation are even worse (b/c of zero close to origin).