ECE 3510 Discrete-time Systems & Transfer Functions

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4/20/20

Section 6.4 in Bodson text (p.200) Follow along in the Textbook

Ex.1 (\$ I got in bank) = (\$ I had) + interest + (\$ I add)

Define: $y(k)$ = bank account balance at end of day k

$x(k)$ = money deposited on day k

α = interest earned in one day

$$y(k) = y(k-1) + \alpha \cdot y(k-1) + x(k)$$

$$\mathbf{Y}(z) = z^{-1} \cdot \mathbf{Y}(z) + \alpha \cdot z^{-1} \cdot \mathbf{Y}(z) + \mathbf{X}(z)$$

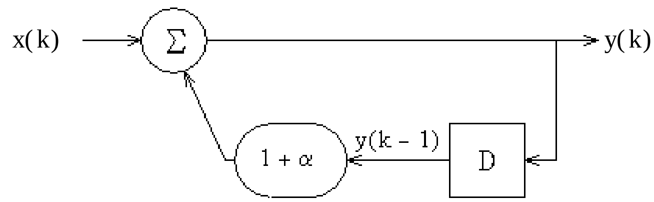
$$= z^{-1} \cdot \mathbf{Y}(z) \cdot (1+\alpha) + \mathbf{X}(z)$$

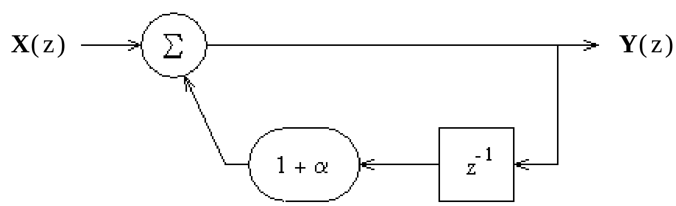
$$\mathbf{Y}(z) - z^{-1} \cdot \mathbf{Y}(z) \cdot (1+\alpha) = \mathbf{X}(z)$$

$$\mathbf{Y}(z) \cdot \left[1 - z^{-1} \cdot (1+\alpha)\right] = \mathbf{X}(z)$$

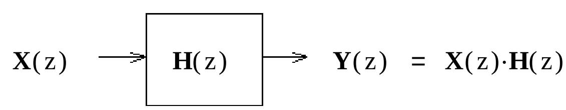
$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)} = \frac{1}{\left[1 - z^{-1} \cdot (1+\alpha)\right]} \cdot \frac{z}{z}$$

$$\mathbf{H}(z) = \frac{z}{z - (1+\alpha)}$$



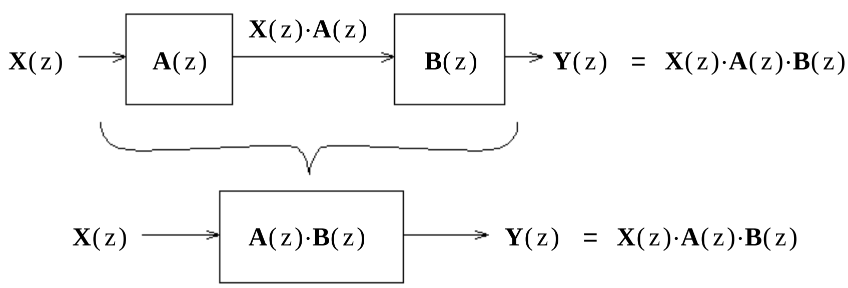


In general: $\mathbf{H}(z) = \dfrac{\text{output}}{\text{input}} = \dfrac{\mathbf{Y}(z)}{\mathbf{X}(z)}$

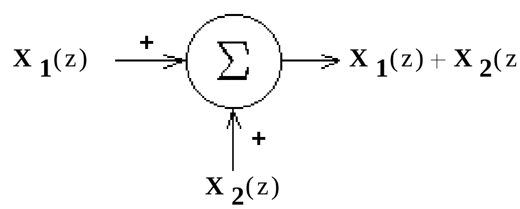


All Transfer - Function and Block - Diagrams we already know from Laplace work with z-transforms

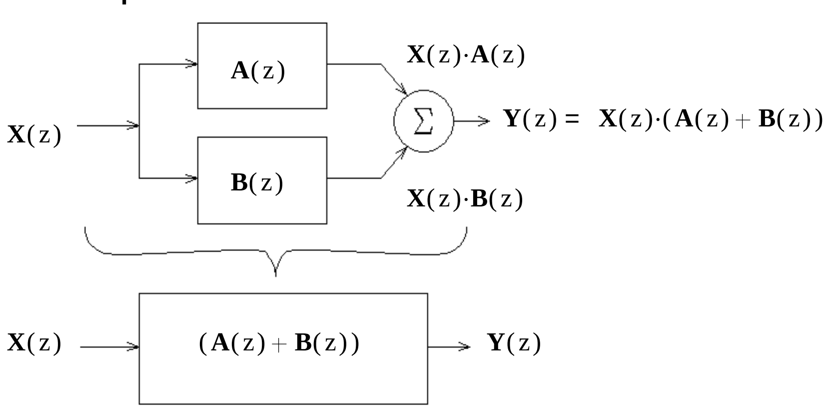
Serial - path systems



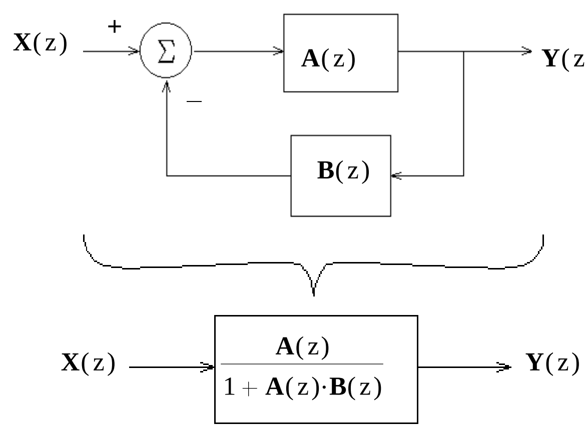
Summers



Parallel - paths



Feedback loop



Impulse Response

$\mathbf{X}(z) = 1 \longrightarrow$ [$\mathbf{H}(z)$] $\longrightarrow \mathbf{Y}(z) = \mathbf{X}(z)\cdot\mathbf{H}(z) = 1\cdot\mathbf{H}(z) = \mathbf{H}(z)$ = Impulse response = z - transform of $h(k)$

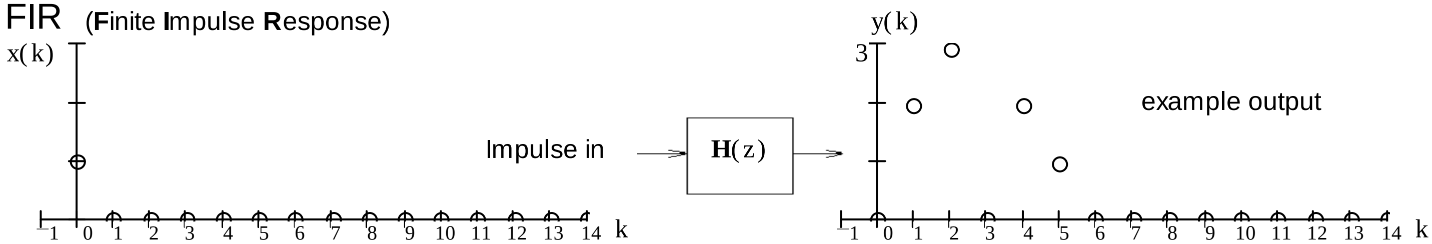
Input is an impulse $\quad y(k) = h(k)$

Sometimes the term "impulse response" is used in place of the term "transfer function"

FIR **F**inite **I**mpulse **R**esponse (FIR) means that output goes to and stays at absolute 0 within a finite number of steps.

IIR **I**nfinite **I**mpulse **R**esponse (IIR) means output never completely goes away. (It may approach 0 like exponential decay)

FIR (**F**inite **I**mpulse **R**esponse)



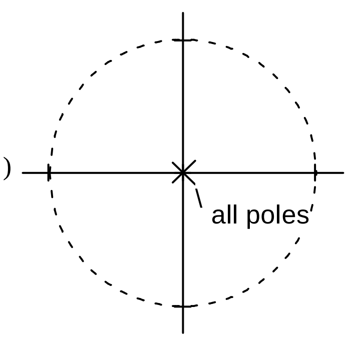
$$y(k) = 2\cdot x(k-1) + 3\cdot x(k-2) + 2\cdot x(k-4) + x(k-5)$$

$$\mathbf{Y}(z) = 2\cdot z^{-1}\cdot\mathbf{X}(z) + 3\cdot z^{-2}\cdot\mathbf{X}(z) + 2\cdot z^{-4}\cdot\mathbf{X}(z) + z^{-5}\cdot\mathbf{X}(z)$$

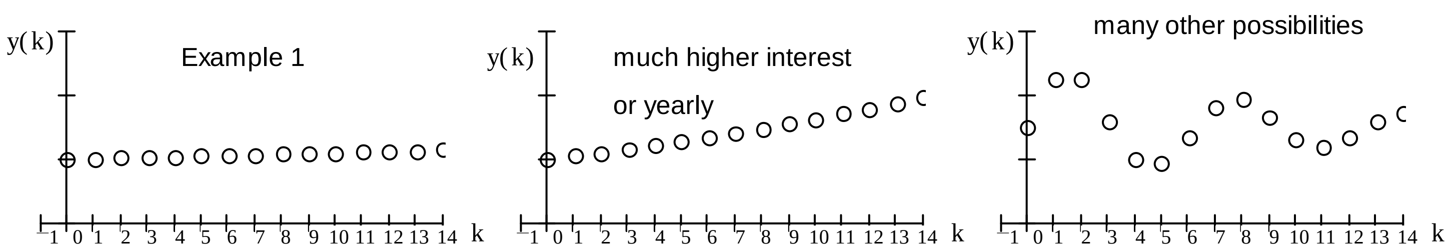
$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)} = \left(2\cdot z^{-1} + 3\cdot z^{-2} + 2\cdot z^{-4} + z^{-5}\right)\cdot\frac{z^5}{z^5}$$

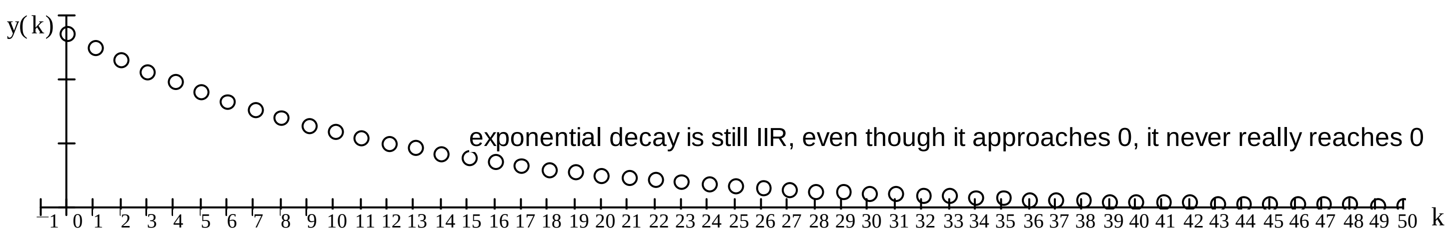
$$\mathbf{H}(z) = \frac{2\cdot z^4 + 3\cdot z^3 + 2\cdot z + 1}{z^5}$$

Poles of $\mathbf{H}(z)$ all at origin



IIR (**I**nfinite **I**mpulse **R**esponse)





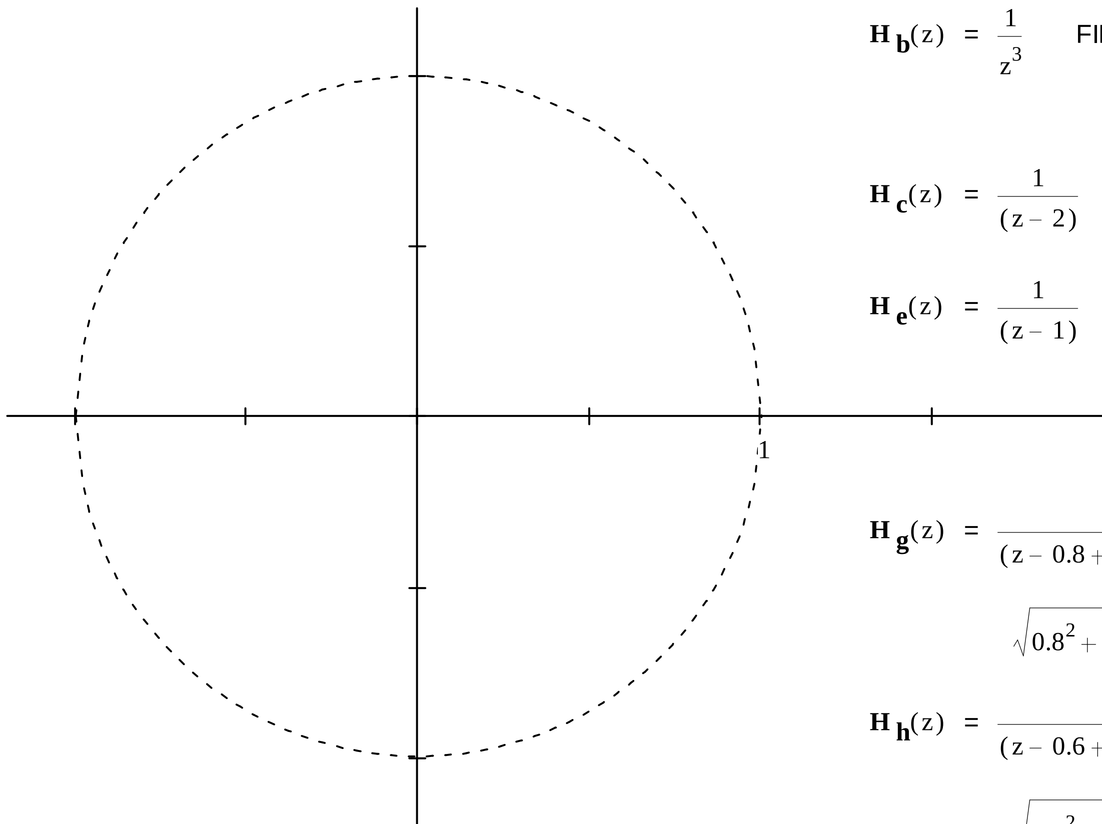
Bounded-Input, Bounded-Output (BIBO) Stable

A system is considered BIBO stable if the output in bounded for any bounded input.

A bounded input could have single poles on the unit circle at any location.
A bounded output may not have double poles on the unit circle or any poles outside the unit circle.
The output will have all the poles of the input plus all the poles of the system. (except in rare pole-zero cancellations.)

Therefore: A BIBO system may not have any poles on the or outside the unit circle.

Draw the poles on this unit circle



$$H_a(z) = \frac{1}{z\cdot(z-0.5)}$$

$$H_b(z) = \frac{1}{z^3}$$ FIR

$$H_c(z) = \frac{1}{(z-2)} \qquad H_d(z) = \frac{1}{(z+2)}$$

$$H_e(z) = \frac{1}{(z-1)} \qquad H_f(z) = \frac{1}{(z+1)}$$

$$H_g(z) = \frac{1}{(z-0.8+0.8\cdot j)\cdot(z-0.8-0.8\cdot j)}$$

$$\sqrt{0.8^2+0.8^2} = 1.131 = |p|$$

$$H_h(z) = \frac{1}{(z-0.6+0.8\cdot j)\cdot(z-0.6-0.8\cdot j)}$$

$$\sqrt{0.6^2+0.8^2} = 1 = |p|$$

$$H_i(z) = \frac{1}{(z-0.6+0.6\cdot j)\cdot(z-0.6-0.6\cdot j)}$$

$$\sqrt{0.6^2+0.6^2} = 0.849 = |p|$$

a,b, YES inside unit circle

c,d, NO outide

e,f, NO right on unit circle

g, NO outide

h, NO right on unit circle

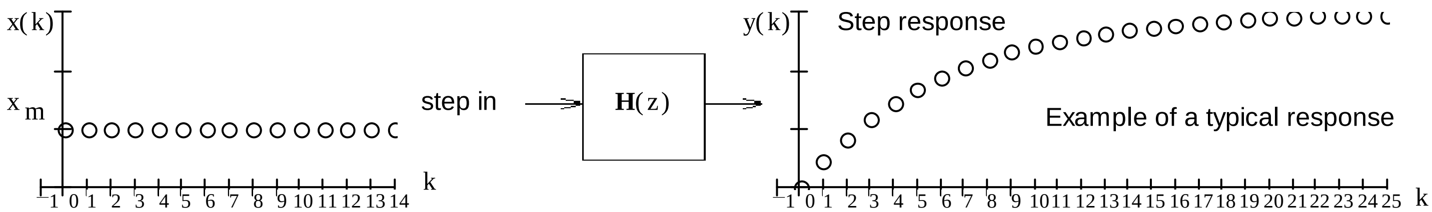
i, YES inside unit circle

Step Response

Remember: Continuous-time (Laplace) $\mathbf{Y}_{ss}(s) = \dfrac{X_m \cdot \mathbf{H}(0)}{z}$ $\quad y_{ss}(t) = X_m \cdot \mathbf{H}(0) \cdot u(t)$ $\quad H(0)$ = DC Gain

Sooo.. yesterday

Today . . .



$\mathbf{X}(z) = X_m \cdot u(k)$

Steady-State Response & DC Gain

For BIBO Systems

$$\mathbf{Y}(z) = \mathbf{X}(z) \cdot \mathbf{H}(z)$$

Complete step response = steady-state response + transient response

partial fraction expansion: $\mathbf{Y}(z) = X_m \cdot \dfrac{z}{z-1} \cdot \mathbf{H}(z) = A + \left[\dfrac{B}{(\blacksquare)} + \dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)}\right]$

divide both sides by z: $\dfrac{\mathbf{Y}(z)}{z} = X_m \cdot \dfrac{1}{z-1} \cdot \mathbf{H}(z) = \dfrac{A}{z-1} + \left[\dfrac{B}{(\blacksquare)} + \dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)}\right] \cdot \dfrac{1}{z}$

multiply both sides by $(z-1)$: $\mathbf{Y}(z) \cdot \dfrac{z-1}{z} = X_m \cdot \mathbf{H}(z) = A + \left[\dfrac{B}{(\blacksquare)} + \dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)}\right] \cdot \dfrac{z-1}{z}$

set $z := 1$: $X_m \cdot \mathbf{H}(1) = A$

$$\mathbf{Y}(z) = X_m \cdot \frac{z}{z-1} \cdot \mathbf{H}(z) = X_m \cdot \mathbf{H}(1) \cdot \frac{z}{z-1} + \left[\frac{B}{(\blacksquare)} + \frac{C}{(\blacksquare)} + \frac{D}{(\blacksquare)}\right]$$

steady-state response

transient response (all other poles are inside unit circle (BIBO))

$$y_{ss}(k) = X_m \cdot \mathbf{H}(1) \cdot u(k)$$

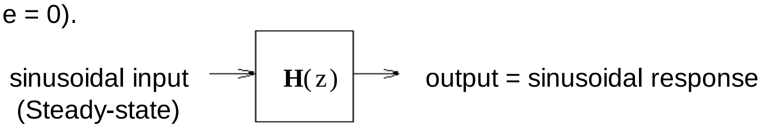
$H(1)$ = DC Gain

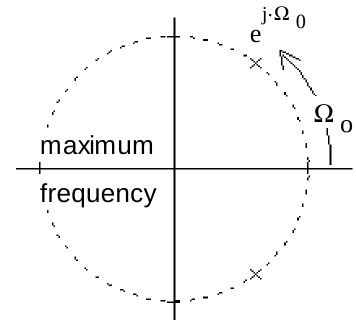
The **transient part** would be found by finishing the partial-fraction expansion.

Sinusoidal Response

For BIBO Systems

The sinusoidal response of a system is the output when the input is a sinusoidal (which starts at time = 0).





For continuous time, we found $\mathbf{H}(j\omega) = |\mathbf{H}(j\omega)| \angle \mathbf{H}(j\omega)$ all $j\omega$ are on the Imaginary axis

For discrete time, we find $\mathbf{H}(p) = |\mathbf{H}(p)| \angle \mathbf{H}(p)$ where all p are on the unit circle

That means that $p = 1 \angle \Omega_0 = 1\cdot e^{j\cdot\Omega_0} = e^{j\cdot\Omega_0}$

$$\mathbf{H}\left(e^{j\cdot\Omega_0}\right) = \left|\mathbf{H}\left(e^{j\cdot\Omega_0}\right)\right| \angle \mathbf{H}(e^{j\Omega_0})$$

Use in the same way.

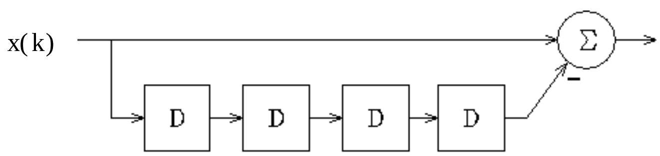
Either:

Modify the magnitude and phase of the input to get the steady-state output, $y_{ss}(k)$ (multiply magnitudes & add phases)

OR $\mathbf{Y}(z) = \mathbf{X}(z)\cdot\mathbf{H}\left(e^{j\cdot\Omega_0}\right)$ Which gives both steady-state and transient outputs.

to get a frequency response plot, allow to vary from 0 (or near 0) to the maximum frequency.

Example from text:



$y(k) = x(k) - x(k-4)$ FIR system

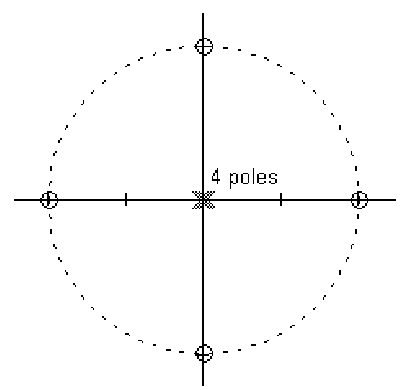
$$\mathbf{Y}(z) = \mathbf{X}(z) - z^{-4}\cdot\mathbf{X}(z) = \mathbf{X}(z)\cdot\left(1 - z^{-4}\right)$$

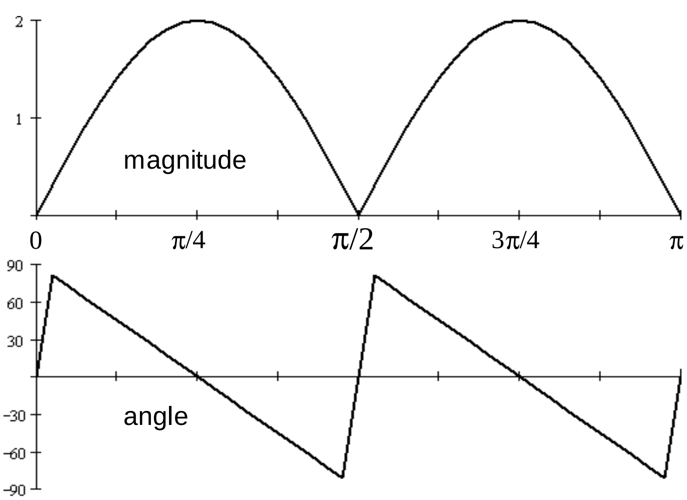
$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)} = 1 - z^{-4} = 1 - \frac{1}{z^4}$$

$$= \frac{z^4}{z^4} - \frac{1}{z^4} = \frac{z^4 - 1}{z^4} = \frac{\left(z^2+1\right)\cdot\left(z^2-1\right)}{z^4}$$

$$\mathbf{H}(z) = \frac{z^4 - 1}{z^4}$$

$$\mathbf{H}\left(e^{j\cdot\Omega_0}\right) = \frac{\left(e^{j\cdot\Omega_0}\right)^4 - 1}{\left(e^{j\cdot\Omega_0}\right)^4} = \frac{e^{j\cdot\Omega_0\cdot 4} - 1}{e^{j\cdot\Omega_0\cdot 4}}$$





These strange, repeating frequency-response curves are common in digital signal processing. Take a class in DSP to learn more. Here, this is about as deep as we're going.

The **transient part** would be found by partial-fraction expansion.

Initial Conditions

Initial Conditions are handled here much like they are in continuous time, with similar results.
In a BIBO system their effects dissappear quickly and are very similar to the impulse response.

Integration

$$y(k) = y(k-1) + x(k)$$ Accumulation

old sum + new

$$\mathbf{Y}(z) = z^{-1}\cdot\mathbf{Y}(z) + \mathbf{X}(z)$$

$$\mathbf{Y}(z) - z^{-1}\cdot\mathbf{Y}(z) = \mathbf{X}(z)$$

$$\mathbf{Y}(z)\cdot\left(1 - z^{-1}\right) = \mathbf{X}(z)$$

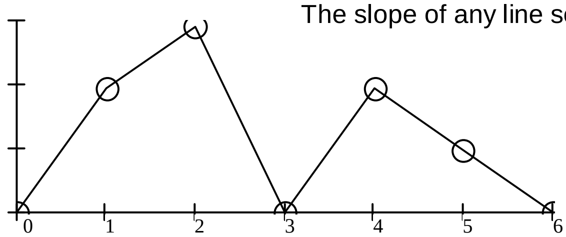
$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

Compare to Laplace, where the transfer function for integration is $\frac{1}{s}$

In both cases this is also the transform of the unit step function.

That's because convolution of a signal with the unit step function is the same as integration.

Differentiation



The slope of any line segment is $y(k) = \frac{\text{rise}}{\text{run}} = \frac{x(k) - x(k-1)}{1}$

$$\mathbf{Y}(z) = \mathbf{X}(z) - z^{-1}\cdot\mathbf{X}(z)$$

$$\mathbf{Y}(z) = \mathbf{X}(z)\cdot\left(1 - z^{-1}\right)$$

$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)} = 1 - z^{-1} = \frac{z-1}{z}$$

Compare to Laplace, where the transfer function for integration is s.

In both cases this is the inverse of transform of integration.

In continuous time, diffential equations play a very important role in describing the world.
In the digital, they become difference equations

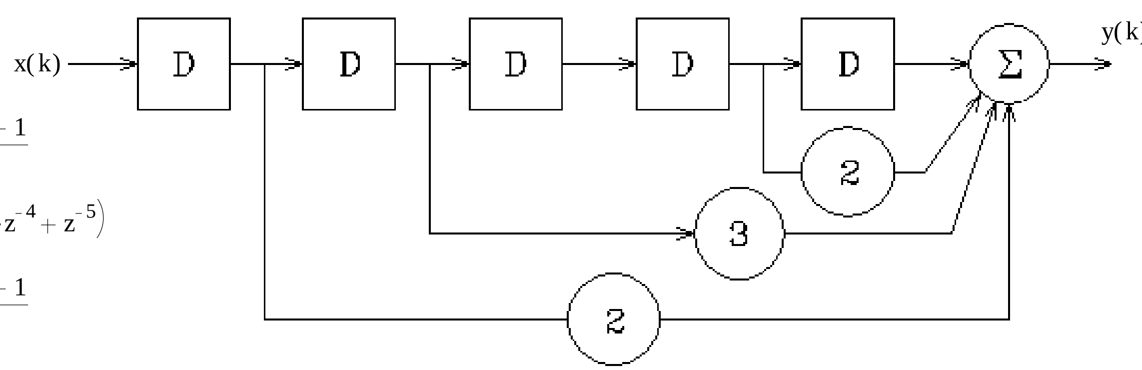
Implementation

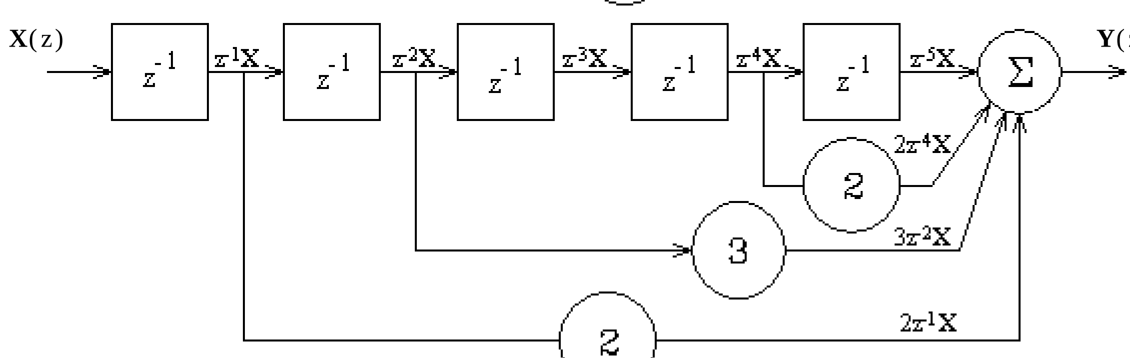
FIR Example:

$$\mathbf{H}(z) = \frac{2\cdot z^4 + 3\cdot z^3 + 2\cdot z + 1}{z^5}$$

$$= \left(2\cdot z^{-1} + 3\cdot z^{-2} + 2\cdot z^{-4} + z^{-5}\right)$$

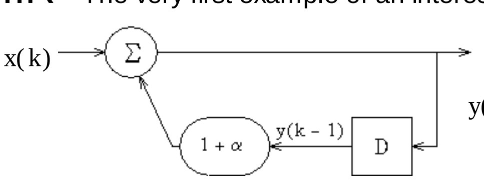
$$= \frac{2\cdot z^4 + 3\cdot z^3 + 2\cdot z + 1}{z^5}$$



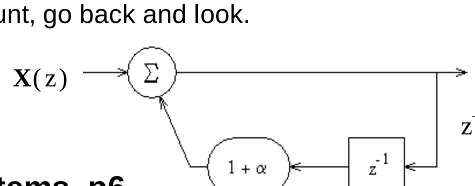


IIR

The very first example of an interest bearing bank account, go back and look.



$$y(k) = y(k-1)\cdot(1+\alpha) + x(k)$$



$$\mathbf{Y}(z) = z^{-1}\cdot\mathbf{Y}(z)\cdot(1+\alpha) + \mathbf{X}(z)$$

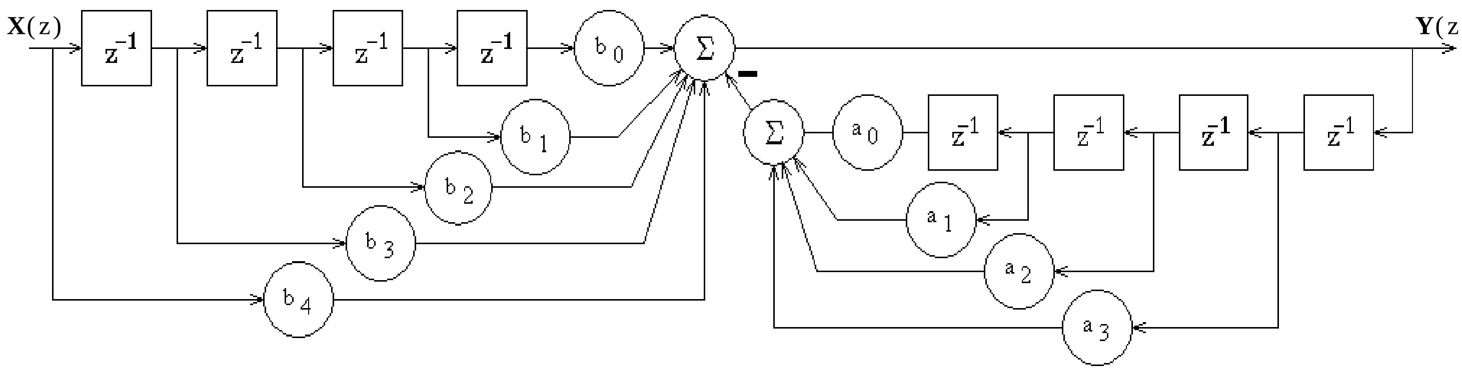
IIR General Example

$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)} = \frac{b_4 \cdot z^4 + b_3 \cdot z^3 + b_2 \cdot z^2 + b_1 \cdot z + b_0}{z^4 + a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0} = \frac{b_4 + b_3 \cdot z^{-1} + b_2 \cdot z^{-2} + b_1 \cdot z^{-3} + b_0 \cdot z^{-4}}{1 + a_3 \cdot z^{-1} + a_2 \cdot z^{-2} + a_1 \cdot z^{-3} + a_0 \cdot z^{-4}}$$

$$\mathbf{Y}(z) \cdot \left(1 + a_3 \cdot z^{-1} + a_2 \cdot z^{-2} + a_1 \cdot z^{-3} + a_0 \cdot z^{-4}\right) = \mathbf{X}(z) \cdot \left(b_4 + b_3 \cdot z^{-1} + b_2 \cdot z^{-2} + b_1 \cdot z^{-3} + b_0 \cdot z^{-4}\right)$$

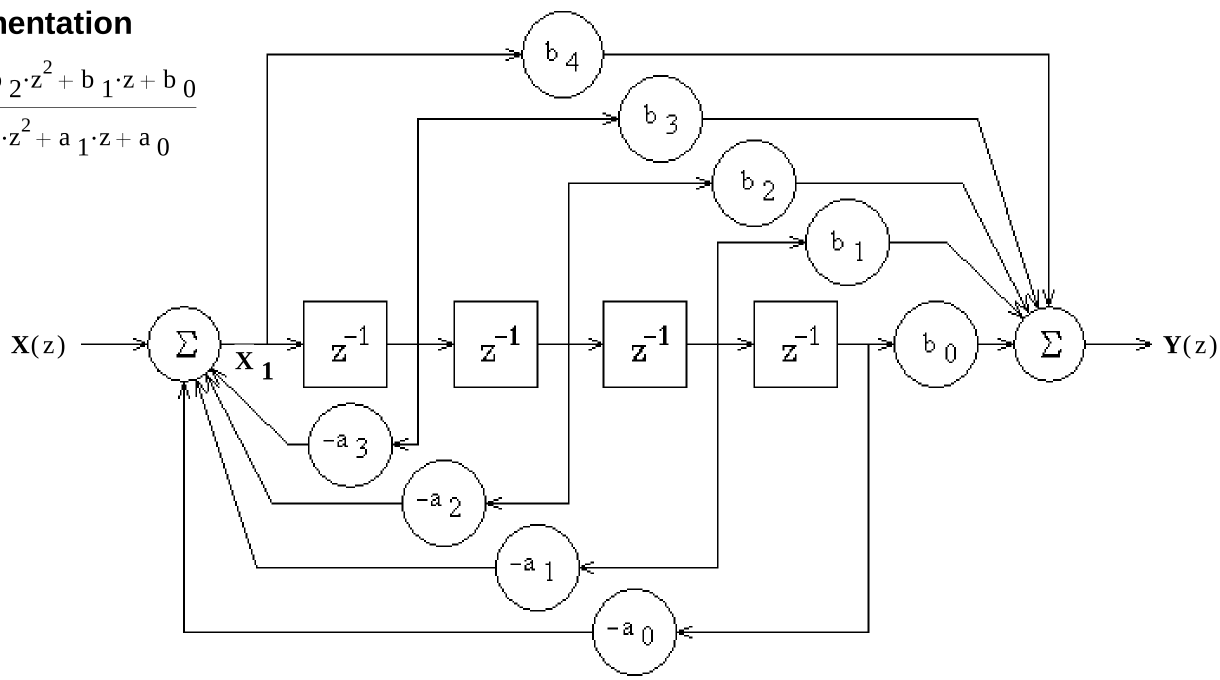
$$\mathbf{Y}(z) = \mathbf{X}(z) \cdot \left(b_4 + b_3 \cdot z^{-1} + b_2 \cdot z^{-2} + b_1 \cdot z^{-3} + b_0 \cdot z^{-4}\right) - \mathbf{Y}(z) \cdot \left(a_3 \cdot z^{-1} + a_2 \cdot z^{-2} + a_1 \cdot z^{-3} + a_0 \cdot z^{-4}\right)$$

Direct Implementation



Minimal Implementation

$$\frac{b_4 \cdot z^4 + b_3 \cdot z^3 + b_2 \cdot z^2 + b_1 \cdot z + b_0}{z^4 + a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0}$$



$$\mathbf{X_1} = \mathbf{X}(z) - a_3 \cdot z^{-1} \cdot \mathbf{X_1} - a_2 \cdot z^{-2} \cdot \mathbf{X_1} - a_1 \cdot z^{-3} \cdot \mathbf{X_1} - a_0 \cdot z^{-4} \cdot \mathbf{X_1}$$

$$\mathbf{X_1} \cdot \left(1 + a_3 \cdot z^{-1} + a_2 \cdot z^{-2} + a_1 \cdot z^{-3} + a_0 \cdot z^{-4}\right) = \mathbf{X}(z)$$

$$\mathbf{X_1} = \frac{\mathbf{X}}{1 + a_3 \cdot z^{-1} + a_2 \cdot z^{-2} + a_1 \cdot z^{-3} + a_0 \cdot z^{-4}}$$

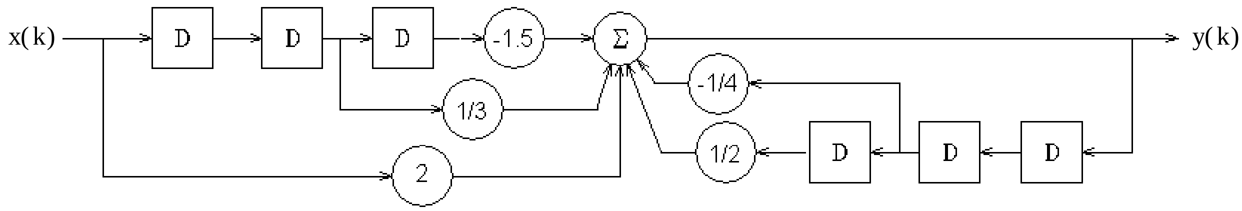
$$\mathbf{Y}(z) = \mathbf{X_1} \cdot \left(b_4 + b_3 \cdot z^{-1} + b_2 \cdot z^{-2} + b_1 \cdot z^{-3} + b_0 \cdot z^{-4}\right)$$

$$\mathbf{Y}(z) = \frac{\mathbf{X}}{1 + a_3 \cdot z^{-1} + a_2 \cdot z^{-2} + a_1 \cdot z^{-3} + a_0 \cdot z^{-4}} \cdot \left(b_4 + b_3 \cdot z^{-1} + b_2 \cdot z^{-2} + b_1 \cdot z^{-3} + b_0 \cdot z^{-4}\right)$$

$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)} = \frac{b_4 + b_3 \cdot z^{-1} + b_2 \cdot z^{-2} + b_1 \cdot z^{-3} + b_0 \cdot z^{-4}}{1 + a_3 \cdot z^{-1} + a_2 \cdot z^{-2} + a_1 \cdot z^{-3} + a_0 \cdot z^{-4}}$$ Check, it works

Example From Spring 2011 Final   a) Draw the block diagram of a simple direct implementation of the difference equation.

$$y(k) = 2\cdot x(k) + \frac{x(k-2)}{3} - 1.5\cdot x(k-3) - \frac{1}{4}\cdot y(k-2) + \frac{1}{2}\cdot y(k-3)$$



b) Find the $\mathbf{H}(z)$ corresponding to the difference equation above. Show your work.

$$\mathbf{Y}(z) = 2\cdot\mathbf{X}(z) + \frac{1}{3}\cdot\mathbf{X}(z)\cdot z^{-2} - 1.5\cdot\mathbf{X}(z)\cdot z^{-3} - \frac{1}{4}\cdot\mathbf{Y}(z)\cdot z^{-1} + \frac{1}{2}\cdot\mathbf{Y}(z)\cdot z^{-2}$$

$$\mathbf{Y}(z) + \frac{1}{4}\cdot\mathbf{Y}(z)\cdot z^{-2} - \frac{1}{2}\cdot\mathbf{Y}(z)\cdot z^{-3} = 2\cdot\mathbf{X}(z) + \frac{1}{3}\cdot\mathbf{X}(z)\cdot z^{-2} - 1.5\cdot\mathbf{X}(z)\cdot z^{-3}$$

$$\mathbf{Y}(z)\cdot\left(1 + \frac{1}{4}\cdot z^{-2} - \frac{1}{2}\cdot z^{-3}\right) = \mathbf{X}(z)\cdot\left(2 + \frac{1}{3}\cdot z^{-2} - 1.5\cdot z^{-3}\right)$$

$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)} = \frac{2 + \frac{1}{3}\cdot z^{-2} - 1.5\cdot z^{-3}}{1 + \frac{1}{4}\cdot z^{-2} - \frac{1}{2}\cdot z^{-3}}\cdot\left(\frac{z^3}{z^3}\right) = \frac{2\cdot z^3 + \frac{1}{3}\cdot z - 1.5}{z^3 + \frac{1}{4}\cdot z - \frac{1}{2}}$$

c) List the poles of $\mathbf{H}(z)$. Indicate multiple poles if there are any.

$$0 = z^3 + \frac{1}{4}\cdot z - \frac{1}{2} \quad \text{solves to} \quad \begin{pmatrix} 0.689 \\ -0.345 + 0.779\cdot j \\ -0.345 - 0.779\cdot j \end{pmatrix}$$

Poles at: 0.689, $-0.345 + 0.779\cdot j$, $-0.345 - 0.779\cdot j$

d) Is this system BIBO stable? Justify your answer.

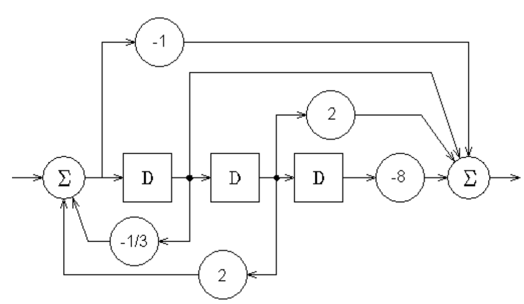
Yes, all poles are inside the unit circle

$0.689 < 1$ $\sqrt{0.345^2 + 0.779^2} = 0.852 < 1$

Another Example from the same Final

Draw a minimal implementation of a system with the following transfer function

$$\mathbf{H}(z) = \frac{-z^3 + (z-2)\cdot(z+4)}{z\cdot\left(z^2 + \frac{z}{3} - 2\right)} \quad \text{find} \quad = \frac{-z^3 + z^2 + 2\cdot z - 8}{z^3 + \frac{1}{3}\cdot z^2 - 2\cdot z}$$

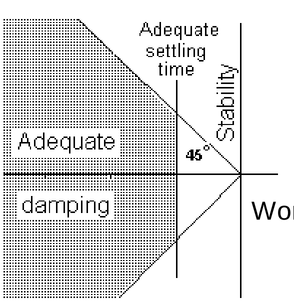


Continuous Time

Differential Equations

Laplace Transform

- Left-half plane / Right-half plane
- Origin
- Frequency increases as pole goes up, vertically



Discrete Time

Difference Equations

z transform

- Inside unit circle / outside unit circle
- Point at (1,0), the right-most point on unit circle
- Frequency increases as pole goes around unit circle
- Extra z in numerator of most terms
  - Divide by z before partial-fraction expansion

Transfer functions and Block diagrams

Same   Lots of z^{-1} blocks

Root Locus

Works exactly the same way, but results are interpreted very differently.