ECE 3510 Effect of initial conditions

See Bodson text, section 3.5

Best explained by example, IF: $\mathbf{H}(s) = \dfrac{b_2 \cdot s^2 + b_1 \cdot s + b_0}{s^2 + a_1 \cdot s + a_0} \cdot \mathbf{X}(s)$

We would normally say: $\mathbf{Y}(s) = \dfrac{b_2 \cdot s^2 + b_1 \cdot s + b_0}{s^2 + a_1 \cdot s + a_0} \cdot \mathbf{X}(s)$

But that ignores initial conditions. So let's deconstruct and then reconstruct with initial conditions included.

$$\mathbf{Y}(s) \cdot \left(s^2 + a_1 \cdot s + a_0\right) = \left(b_2 \cdot s^2 + b_1 \cdot s + b_0\right) \cdot \mathbf{X}(s)$$

$$s^2 \cdot \mathbf{Y}(s) + a_1 \cdot s \cdot \mathbf{Y}(s) + a_0 \cdot \mathbf{Y}(s) = b_2 \cdot s^2 \cdot \mathbf{X}(s) + b_1 \cdot s \cdot \mathbf{X}(s) + b_0 \cdot \mathbf{X}(s)$$

$$\frac{d^2}{dt^2} y(t) + a_1 \cdot \frac{d}{dt} y(t) + a_0 \cdot y(t) = b_2 \cdot \frac{d^2}{dt^2} x(t) + b_1 \cdot \frac{d}{dt} x(t) + b_0 \cdot x(t)$$

Laplace Properties

Operation	f(t)	F(s)
Time differentiation	$\frac{d}{dt} f(t)$	$s \cdot \mathbf{F}(s) - f(0-)$
	$\frac{d^2}{dt^2} f(t)$	$s^2 \cdot \mathbf{F}(s) - s \cdot f(0-) - \frac{d}{dt} f(0-)$ initial slope

$$\frac{d^2}{dt^2} y(t) + a_1 \cdot \frac{d}{dt} y(t) + a_0 \cdot y(t) = b_2 \cdot \frac{d^2}{dt^2} x(t) + b_1 \cdot \frac{d}{dt} x(t) + b_0 \cdot x(t)$$

$$\left(s^2 \cdot \mathbf{Y}(s) - s \cdot y(0-) - \frac{d}{dt} y(0-)\right) + a_1 \cdot \left(s \cdot \mathbf{Y}(s) - y(0-)\right) + a_0 \cdot \mathbf{Y}(s) =$$

$$b_2 \cdot \left(s^2 \cdot \mathbf{X}(s) - s \cdot x(0-) - \frac{d}{dt} x(0-)\right) + b_1 \cdot \left(s \cdot \mathbf{X}(s) - x(0-)\right) + b_0 \cdot \mathbf{X}(s)$$

$$\left(s^2 \cdot \mathbf{Y}(s) + a_1 \cdot s \cdot \mathbf{Y}(s) + a_0 \cdot \mathbf{Y}(s)\right) - s \cdot y(0-) - \frac{d}{dt} y(0-) - a_1 \cdot y(0-) =$$

$$\left(b_2 \cdot s^2 \cdot \mathbf{X}(s) + b_1 \cdot s \cdot \mathbf{X}(s) + b_0 \cdot \mathbf{X}(s)\right) - b_2 \cdot s \cdot x(0-) - b_2 \cdot \frac{d}{dt} x(0-) - b_1 \cdot x(0-)$$

$$\mathbf{Y}(s) \cdot \left(s^2 + a_1 \cdot s + a_0\right) = \left(b_2 \cdot s^2 + b_1 \cdot s + b_0\right) \cdot \mathbf{X}(s) + \left(s \cdot y(0-) + \frac{d}{dt} y(0-) + a_1 \cdot y(0-) - b_2 \cdot s \cdot x(0-) - b_2 \cdot \frac{d}{dt} x(0-) - b_1 \cdot x(0-)\right)$$

Response to input | Response to initial conditions

$$\mathbf{Y}(s) = \underbrace{\frac{b_2 \cdot s^2 + b_1 \cdot s + b_0}{s^2 + a_1 \cdot s + a_0}}_{\mathbf{H}(s)} \cdot \mathbf{X}(s) + \frac{s \cdot y(0-) + \frac{d}{dt} y(0-) + a_1 \cdot y(0-) - b_2 \cdot s \cdot x(0-) - b_2 \cdot \frac{d}{dt} x(0-) - b_1 \cdot x(0-)}{s^2 + a_1 \cdot s + a_0}$$

Forced response | Natural response

Zero-state response | Zero-input response

$$\mathbf{Y}(s) = \overbrace{\frac{b_2 \cdot s^2 + b_1 \cdot s + b_0}{s^2 + a_1 \cdot s + a_0}}^{\mathbf{H}(s)} \cdot \mathbf{X}(s) \quad + \quad \overbrace{\frac{s \cdot y(0-) + \frac{d}{dt}y(0-) + a_1 \cdot y(0-) - b_2 \cdot s \cdot x(0-) - b_2 \cdot \frac{d}{dt}x(0-) - b_1 \cdot x(0-)}{s^2 + a_1 \cdot s + a_0}}^{\text{Initial conditions}}$$

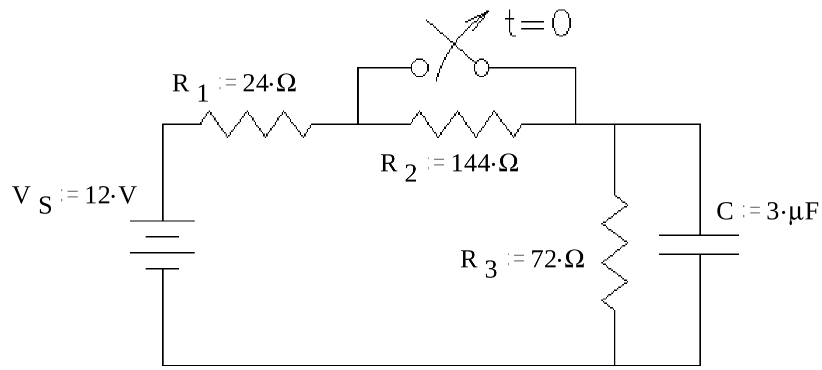
Observations

1. The total response is the sum of two independent components.
2. These values together fully describe the *state* of the 2nd-order system at time $t = 0^-$ (the initial state): $y(0-)$ $\quad \frac{d}{dt}y(0-)$ $\quad x(0-)$ $\quad \frac{d}{dt}x(0-)$
3. Similar denominator for both parts = Share poles = Similar responses
4. Response to Initial conditions always go to zero if system is BIBO.
5. Pole-zero cancellations in right-half plane can cause major problems with internal states of the system.

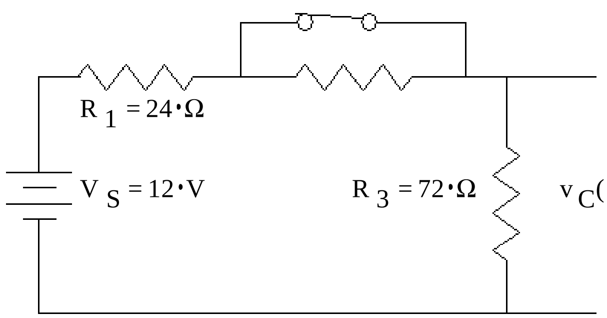
A simple first-order example

The switch has been closed for a long time and is opened at time $t=0$.

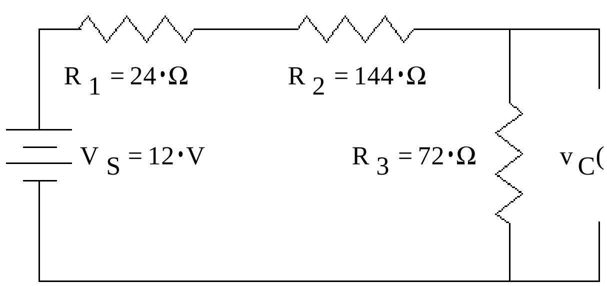
Find the complete expression for $v_C(t)$.



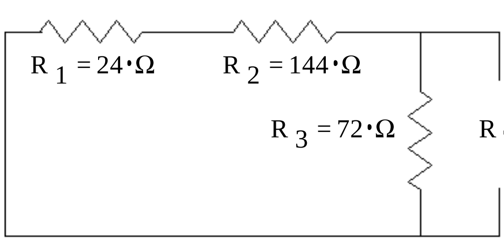
The way you've solved it before:



$$v_C(0) = V_S \cdot \frac{R_3}{R_1 + R_3} = 9 \cdot V$$



$$v_C(\infty) = V_S \cdot \frac{R_3}{R_1 + R_2 + R_3} = 3.6 \cdot V$$



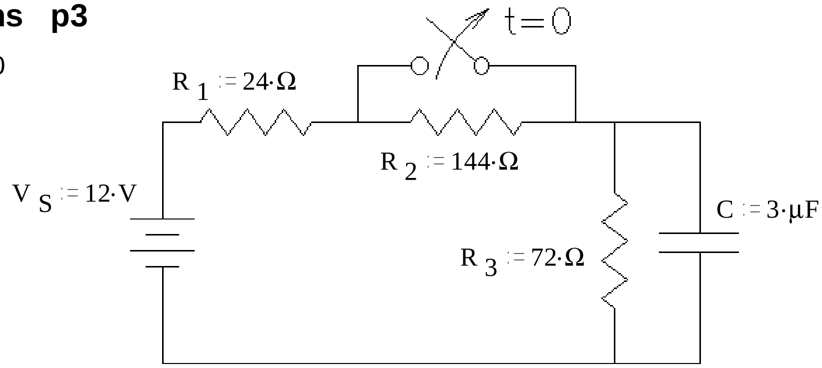
$$R_{Th} := \frac{1}{\left(\frac{1}{R_3} + \frac{1}{R_1 + R_2}\right)} \qquad R_{Th} = 50.4 \cdot \Omega$$

$$\tau := R_{Th} \cdot C \qquad \tau = 151.2 \cdot \mu s$$

$$v_C(t) = v_C(\infty) + \left(v_C(0) - v_C(\infty)\right) \cdot e^{-\frac{t}{\tau}} = 3.6 \cdot V + (9 \cdot V - 3.6 \cdot V) \cdot e^{-\frac{t}{151 \cdot \mu s}} = 3.6 \cdot V + 5.4 \cdot V \cdot e^{-\frac{t}{151 \cdot \mu s}}$$

ECE 3510 Effect of initial conditions p3

The way we would do the same thing in 3510



$$\mathbf{H}(s) = \frac{\mathbf{V}_C(s)}{\mathbf{V}_S(s)} = \frac{\dfrac{1}{\dfrac{1}{R_3}+C\cdot s}}{R_1+R_2+\dfrac{1}{\dfrac{1}{R_3}+C\cdot s}}\cdot\frac{\left(\dfrac{1}{R_3}+C\cdot s\right)}{\left(\dfrac{1}{R_3}+C\cdot s\right)} = \frac{1}{\dfrac{R_1+R_2}{R_3}+(R_1+R_2)\cdot C\cdot s+1}\cdot\frac{\left[\dfrac{1}{(R_1+R_2)\cdot C}\right]}{\left[\dfrac{1}{(R_1+R_2)\cdot C}\right]}$$

$$= \frac{1}{\dfrac{R_1+R_2}{R_3}+(R_1+R_2)\cdot C\cdot s+1}\cdot\frac{\left[\dfrac{1}{(R_1+R_2)\cdot C}\right]}{\left[\dfrac{1}{(R_1+R_2)\cdot C}\right]} = \frac{\dfrac{1}{(R_1+R_2)\cdot C}}{s+\left[\dfrac{1}{R_3\cdot C}+\dfrac{1}{(R_1+R_2)\cdot C}\right]}$$

First-order version of $\mathbf{Y}(s)$ with initial conditions

$$\mathbf{Y}(s) = \underbrace{\frac{b_1\cdot s+b_0}{s+a_0}}_{\mathbf{H}(s)}\cdot\mathbf{X}(s) + \underbrace{\frac{y(0-)-b_1\cdot x(0-)}{s+a_0}}_{\text{Initial conditions}}$$

$b_1 = 0$ $\quad b_0 := \dfrac{1}{(R_1+R_2)\cdot C}$ $\quad b_0 = 1984\cdot\text{sec}^{-1}$

$a_0 := \dfrac{1}{R_3\cdot C}+\dfrac{1}{(R_1+R_2)\cdot C}$ $\quad a_0 = 6614\cdot\text{sec}^{-1}$

$$\mathbf{X}(s) = \frac{12\cdot V}{s}$$

$$\mathbf{Y}(s) = \frac{b_0}{s+a_0}\cdot\frac{12\cdot V}{s} + \frac{y(0-)}{s+a_0}$$

$y(0-) = 9\cdot V$
from above

$$\frac{b_0}{s+a_0}\cdot\frac{12\cdot V}{s} = \frac{A}{s} + \frac{B}{s+a_0}$$

$$A = \mathbf{H}(0)\cdot 12\cdot V = \frac{b_0}{a_0}\cdot 12\cdot V = 3.6\cdot V$$

$$B = b_0\cdot\frac{12\cdot V}{s}\Bigg|_{s:=-a_0} = \frac{b_0}{-a_0}\cdot 12\cdot V = -3.6\cdot V$$

$$= \frac{3.6\cdot V}{s} + \frac{-3.6\cdot V}{s+a_0} + \frac{y(0-)}{s+a_0}$$

$$y(t) = v_C(t) = \left(3.6\cdot V - 3.6\cdot V\cdot e^{-a_0\cdot t} + 9\cdot V\cdot e^{-a_0\cdot t}\right)\cdot u(t)$$

Same as above $\quad v_C(t) = 3.6\cdot V + (9\cdot V - 3.6\cdot V)\cdot e^{-\frac{t}{151\cdot\mu s}}$ $\quad$ where $\tau = \dfrac{1}{a_0} = \dfrac{\text{sec}}{6614} = 151.2\cdot\mu s$

ECE 3510 Effect of initial conditions p3

ECE 3510 State Space

A completely different method where all math is done in the time-domain using linear algebra. See Bodson, section 3.6.

$x(t)$ = The state vector (n x 1 matrix)

n = order of the system

$\frac{d}{dt}x(t)$ = Time derivative of the state vector (n x 1 matrix)

$u(t)$ = The input vector (n_u x 1 matrix)

n_u = number of inputs

$y(t)$ = The state vector (n_y x 1 matrix)

n_y = order of the system

A = The system matrix (n x n matrix)

B = The input matrix (n x n_u matrix)

C = The output matrix (n_y x n matrix)

D = The feed-forward matrix (n_y x n_u matrix)

A third-order, 2-input, 2-output system

State Equation: $$\frac{d}{dt}x(t) = A\cdot x(t) + B\cdot u(t)$$

$$\begin{pmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{pmatrix} = \begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{pmatrix} \cdot \begin{pmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{pmatrix} + \begin{pmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{pmatrix} \cdot \begin{pmatrix} \blacksquare \\ \blacksquare \end{pmatrix}$$

Output Equation: $$y(t) = C\cdot x(t) + D\cdot u(t)$$

$$\begin{pmatrix} \blacksquare \\ \blacksquare \end{pmatrix} = \begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{pmatrix} \cdot \begin{pmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{pmatrix} + \begin{pmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{pmatrix} \cdot \begin{pmatrix} \blacksquare \\ \blacksquare \end{pmatrix}$$

A third-order, single-input, single-output system

State Equation: $$\frac{d}{dt}x(t) = A\cdot x(t) + B\cdot u(t)$$

$$\begin{pmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{pmatrix} = \begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{pmatrix} \cdot \begin{pmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{pmatrix} + \begin{pmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{pmatrix} \cdot u(t)$$

Output Equation: $$y(t) = C\cdot x(t) + D\cdot u(t)$$

$$y(t) = \begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \end{pmatrix} \cdot \begin{pmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{pmatrix} + D\cdot u(t)$$

Advantages of the state-space method

Easily handles multiple inputs, multiple outputs and initial conditions

Can be used with nonlinear systems

Can be used with time-varying systems

Reveals unstable systems that have stable transfer functions (pole-zero cancellations). You can determine:

- Controllability: State variables can all be affected by the input
- Observability: State variables are all "observable" from the output

Basis of Optimal control methods

Advantages and disadvantages of the classical frequency-domain method used in this class

Simpler to understand and model interconnected systems.

Rapidly provide stability and transient response information.

Easy to see the effects of varying system parameters to get a good design.

Limited to linear, time-invariant systems or systems that can be approximated as such.