ECE 3510 BIBO System Responses to an Impulse or a Step input

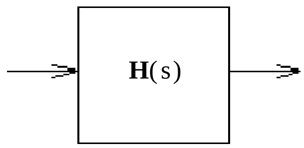
Impulse Response

The Impulse response of a system is the output when the input is an impulse (delta function).

The simplest possible input: $\mathbf{X}(s) = 1$

Input is an impulse

$\mathbf{X}(s) = 1$



output = "impulse response"

(The response of the system to an impulse)

$$\mathbf{Y}(s) = \mathbf{X}(s)\cdot\mathbf{H}(s) = 1\cdot\mathbf{H}(s) = \mathbf{H}(s)$$

A signal who's transform is the system's transfer function

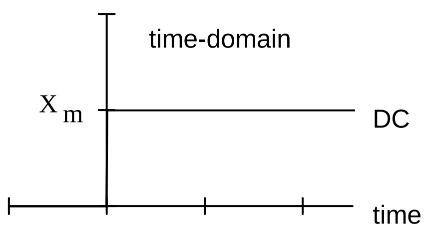
Of course, an impulse is a little impractical in real life.
But, if you can approximate one, than you may be able to use it to characterize an unknown system.

Sometimes the term "impulse response" is used in place of the term "transfer function"

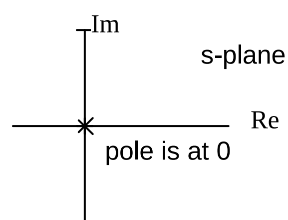
Step Responses

The step response of a system is the output when the input is a step (DC which starts at time-zero).

Step input



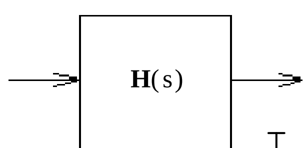
$x(t) = X_m\cdot u(t)$



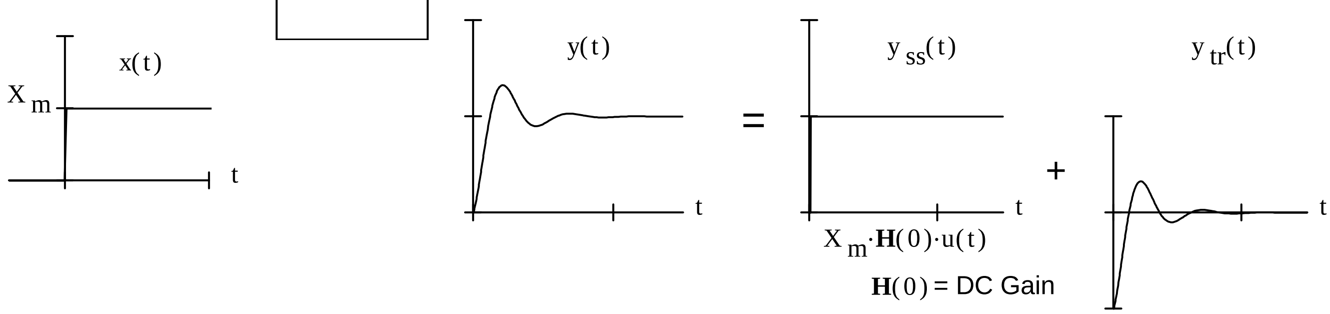
$X(s) = X_m\cdot\frac{1}{s}$

System Step Response

step input



output = step response



$X_m\cdot\mathbf{H}(0)\cdot u(t)$

$\mathbf{H}(0)$ = DC Gain

Complete step response = steady-state response + transient response

Steady-State Response & DC Gain

$$\mathbf{Y}(s) = \mathbf{X}(s)\cdot\mathbf{H}(s) \quad = \frac{X_m}{s}\cdot\mathbf{H}(s)$$

Complete step response

partial fraction expansion: $\mathbf{Y}(s) = \dfrac{X_m}{s}\cdot\mathbf{H}(s) = \dfrac{A}{s} \quad + \quad \dfrac{B}{(\blacksquare)} + \dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)} + \ldots$

steady-state response + transient response

multiply both sides by s $\quad X_m\cdot\mathbf{H}(s) = A \quad + \quad \left[\dfrac{B}{(\blacksquare)} + \dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)}\right]\cdot s$

set $s := 0$ $\quad X_m\cdot\mathbf{H}(0) = A \quad + \quad \left[\dfrac{B}{(\blacksquare)} + \dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)}\right]\cdot 0$

$$\mathbf{Y}_{ss}(s) = \frac{A}{s} = \frac{X_m\cdot\mathbf{H}(0)}{s} \qquad y_{ss}(t) = X_m\cdot\mathbf{H}(0)\cdot u(t)$$

$\mathbf{H}(0)$ = DC Gain

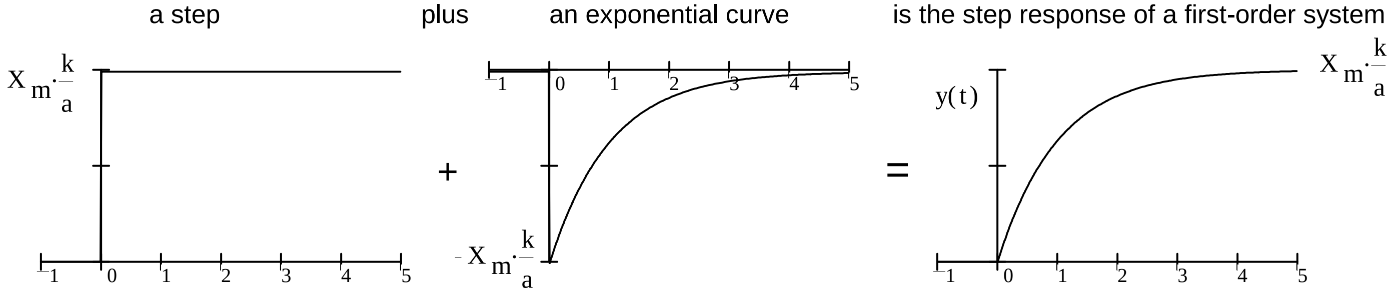
The **transient part** would be found by finishing the partial-fraction expansion.

Step Response of First-Order Systems

$$\mathbf{H}(s) = \frac{k}{s+a} = \frac{k}{s+\frac{1}{\tau}} \qquad \mathbf{Y}(s) = \frac{X_m}{s}\cdot\frac{k}{s+a}$$

$$y(t) = X_m\cdot\left(\frac{k}{a} - \frac{k}{a}\cdot e^{-a\cdot t}\right)\cdot u(t)$$ (ignoring initial conditions)

a step plus an exponential curve is the step response of a first-order system



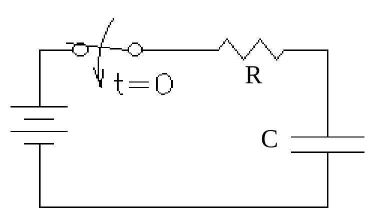
All first-order systems have the same time-domain response:

$$y(t) = y(\infty) + (y(0) - y(\infty))\cdot e^{-\frac{t}{\tau}}$$

$y(0)$ = the initial condition

$y(\infty)$ = the final condition

A simple example of a first-order system

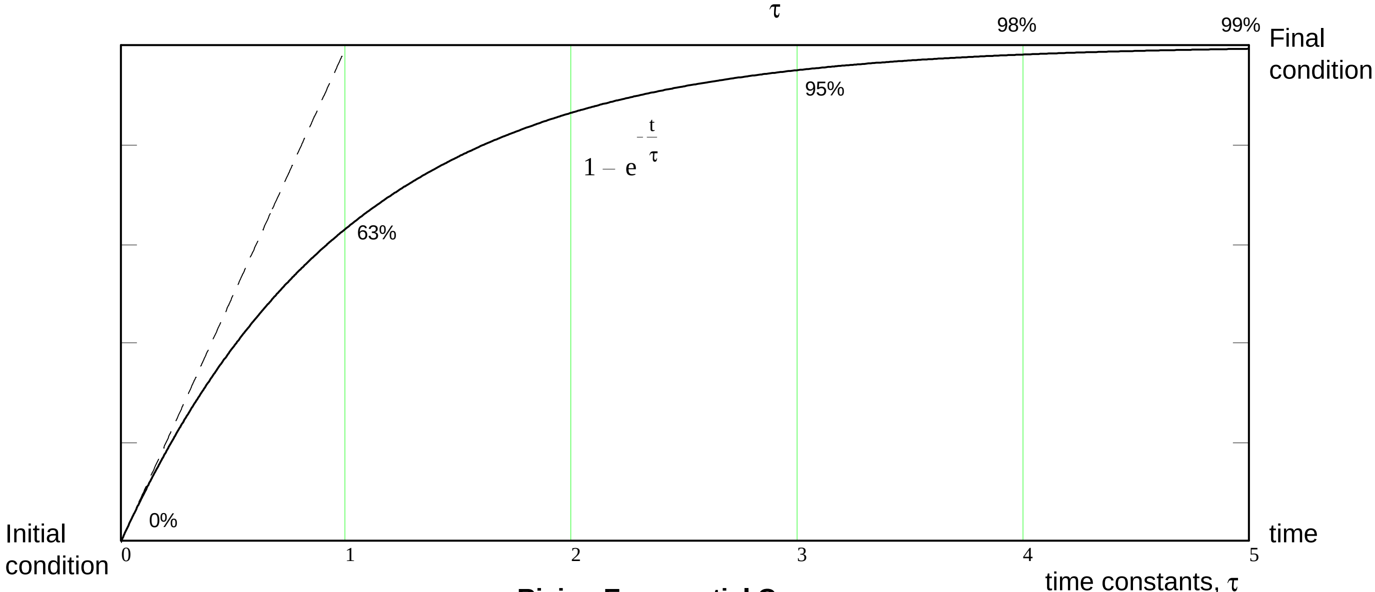


$$v_C(t) = v_C(\infty) + \left(v_C(0) - v_C(\infty)\right)\cdot e^{-\frac{t}{\tau}} \qquad \tau = R\cdot C$$

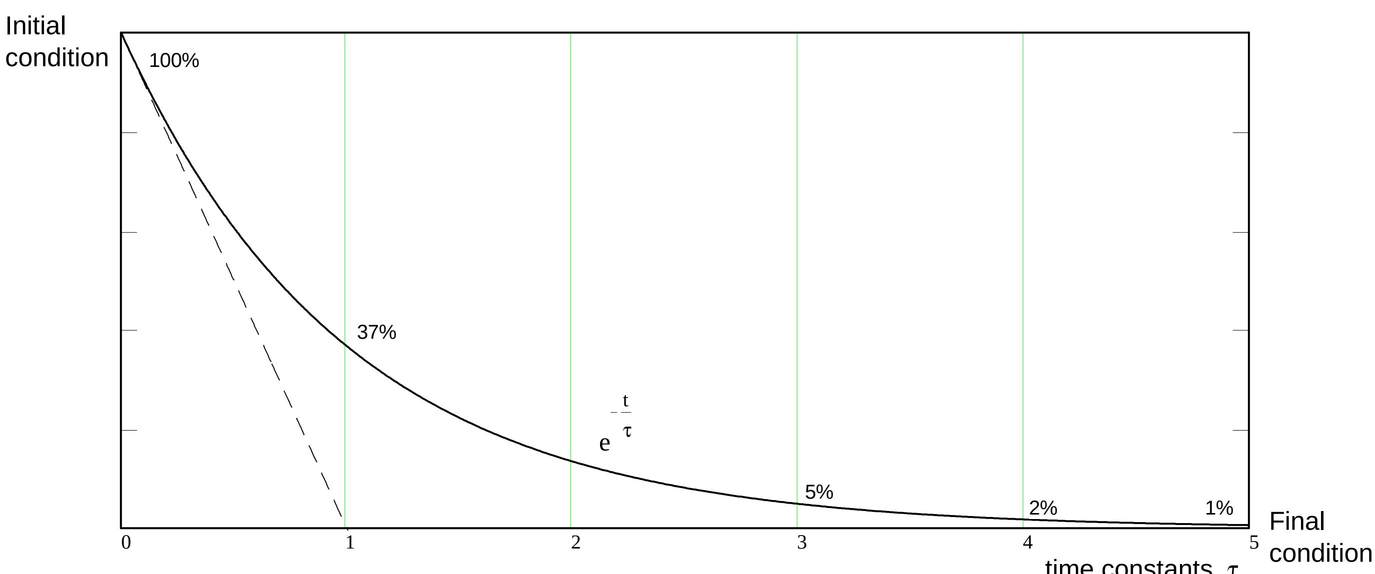
Exponential Curves

Let's take a closer look at some of the characteristics of exponential curves, the output of stable first order system. The transient effects always die out after some time, so the exponents are always negative.

Step response of: $\mathbf{H}(s) = \dfrac{k}{s + \dfrac{1}{\tau}}$



Rising Exponential Curve



Decaying Exponential Curve

Some Important Features:

1) These curves proceed from an initial condition to a final condition. If the final condition is greater than the initial, then the curve is said to be a "rising" exponential. If the final condition is less than the initial, then the curve is called a "decaying" exponential.

2) The curves' initial slope is $\pm$ $1/\tau$. If they continued at this initial slope they'd reach the final condition in one time constant.

3) In the first time constant the curve goes 63% from initial to the final condition.

4) By four time constants the curve is within 2% of the final condition and is usually considered finished. Mathematically, the curve approaches the final condition asymptotically and never reaches it. In reality, of course, this is nonsense. Whatever difference there may be between the mathematical solution and the final condition will soon be overshadowed by random fluctuations (called noise) in the real system.

$e^{-4} = 0.018 \quad < 2\%$

Step Response of Second-Order Systems
Real poles (over and critically damped)

A first-order system for reference

$$\mathbf{H}_1(s) = \frac{k}{s+a} \qquad a := 1 \qquad k := a \qquad y_1(t) := \left(\frac{k}{a} - \frac{k}{a}\cdot e^{-a\cdot t}\right)$$

\ normalization to make curves below easier to compare /

Second-order system, critically damped

$$\mathbf{H}_2(s) = \frac{k}{(s+a)^2} \qquad a := 1 \qquad k := a^2 \qquad y_2(t) := \left(\frac{k}{a^2} - \frac{k}{a^2}\cdot e^{-a\cdot t} - \frac{k}{a}\cdot t\cdot e^{-a\cdot t}\right)$$

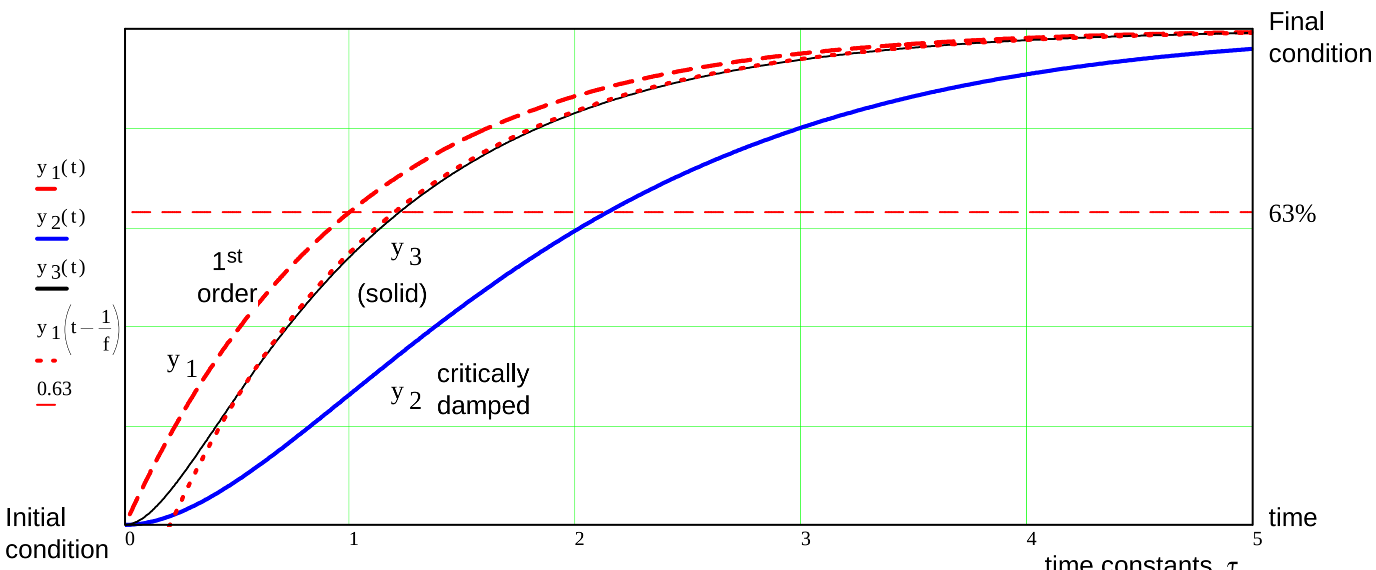
double pole on real axis

Second-order system, over damped

$$\mathbf{H}_3(s) = \frac{k}{(s+a_1)\cdot(s+a_2)}$$

$a_1 := a \qquad f := 5 \qquad a_2 := f\cdot a_1 \qquad k := a_1\cdot a_2$ normalization

$$y_3(t) := \left[\frac{k}{a_1\cdot a_2} + \frac{k}{a_1\cdot(a_1 - a_2)}\cdot e^{-a_1\cdot t} + \frac{k}{a_2\cdot(a_2 - a_1)}\cdot e^{-a_2\cdot t}\right]$$



Rising Curves

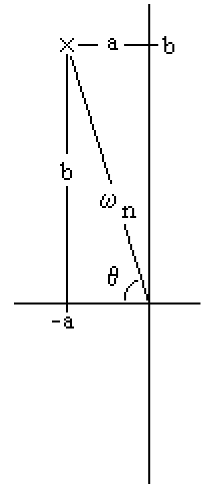
Some Important Features:

1) The poles closest to the jω axis are the **dominant** poles.

2) Poles to the left of the dominant poles may introduce an effect that looks like **time delay**.

3) Conversely, the effects of a time delay (non-linear) can sometimes be modeled by an extra pole (linear) to the left of the dominant poles.

Step Responses of Under-Damped 2nd order Systems (Complex poles)



$$\mathbf{H}(s) = \frac{k}{s^2 + 2\cdot a\cdot s + a^2 + b^2} = \frac{k}{\omega_n{}^2}\cdot\frac{\omega_n{}^2}{s^2 + 2\cdot\zeta\cdot\omega_n\cdot s + \omega_n{}^2}$$

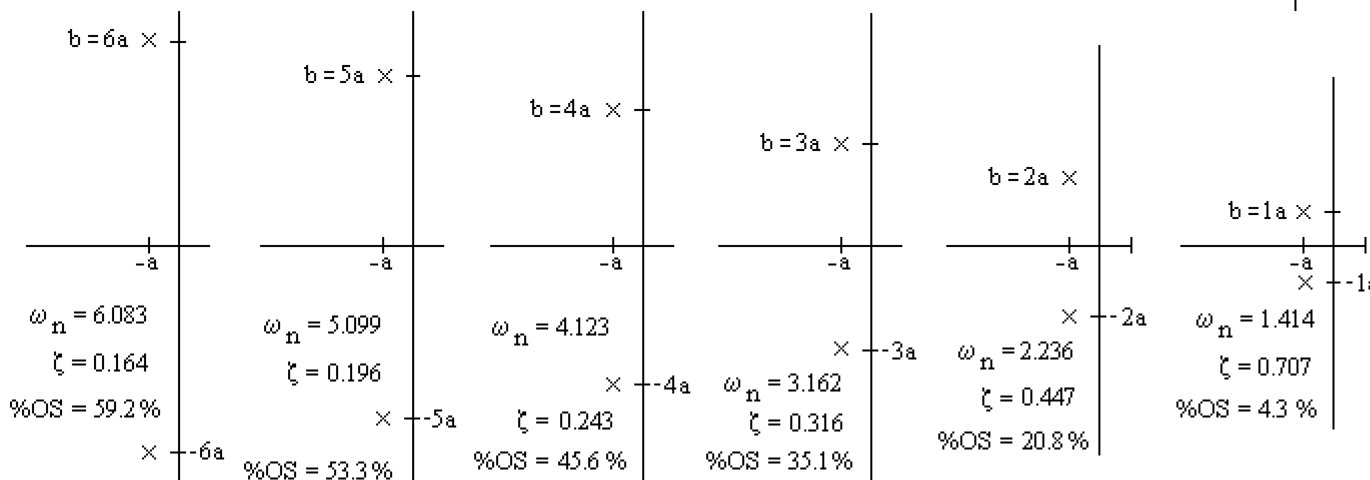
$\omega_n{}^2 = a^2 + b^2$ ω_n = natural frequency

DC gain $\zeta\cdot\omega_n = a$

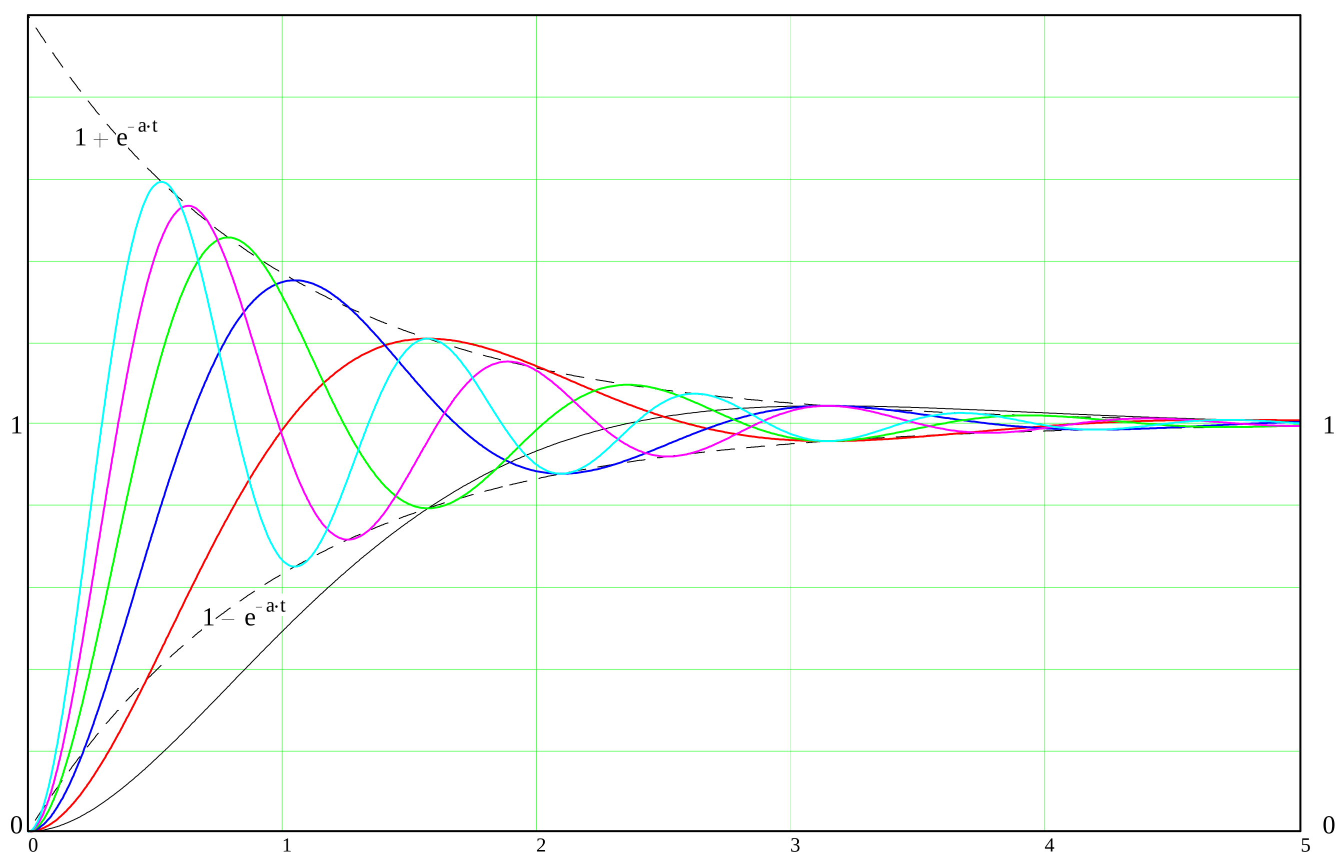
$$\mathbf{H}(0) = \frac{k}{a^2 + b^2} = \frac{k}{\omega_n{}^2} \qquad \zeta = \frac{a}{\omega_n} = \frac{a}{\sqrt{a^2 + b^2}} \text{ = damping factor } = \cos(\theta)$$

$$y(t) = x_m\cdot\mathbf{H}(0)\cdot\left(1 - e^{-a\cdot t}\cdot\cos(b\cdot t) - \frac{a}{b}\cdot e^{-a\cdot t}\cdot\sin(b\cdot t)\right)$$

(curves below are normalized so $\mathbf{H}(0) = 1$)



Overshoot
$\%OS = 100\cdot\%\cdot e^{-\frac{a}{b}\cdot\pi}$



Underdamped Curves

Step Responses, Effect of Zeroes

A first-order system for reference

$$\mathbf{H_1}(s) = \frac{k}{s+a} \qquad a := 1 \qquad k := a \qquad y_1(t) := \left(\frac{k}{a} - \frac{k}{a}\cdot e^{-a\cdot t}\right)$$

An overdamped system with a single zero

$$\mathbf{H}(s) = \frac{k\cdot(s+z)}{(s+a_1)\cdot(s+a_2)} \qquad \mathbf{Y}(s) = \frac{X_m}{s}\cdot\frac{k\cdot(s+z)}{(s+a_1)\cdot(s+a_2)}$$

k is normalized so the curves below will not reach the same final condition.

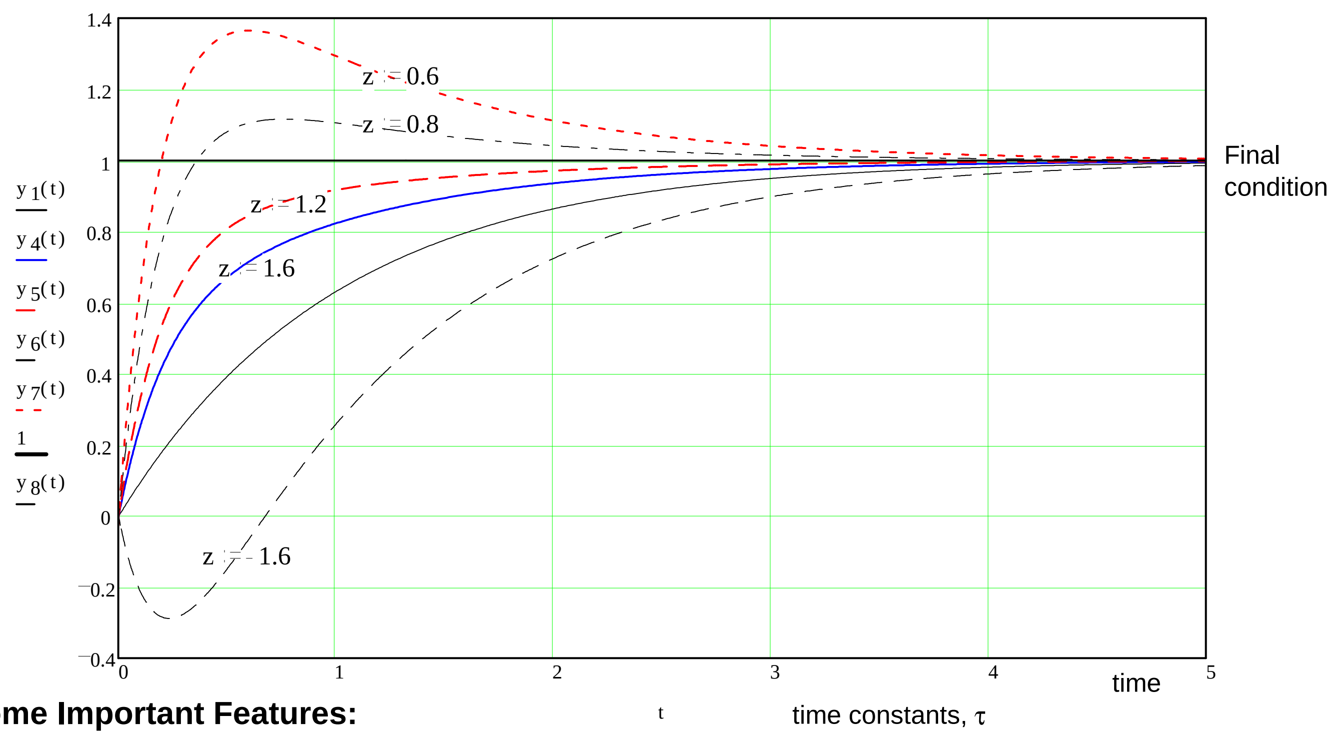
$$z := 1.6 \qquad k := \frac{a_1\cdot a_2}{z} \qquad y_4(t) := \left[\frac{k\cdot z}{a_1\cdot a_2} + \frac{k\cdot(z-a_1)}{a_1\cdot(a_1-a_2)}\cdot e^{-a_1\cdot t} + \frac{k\cdot(z-a_2)}{a_2\cdot(a_2-a_1)}\cdot e^{-a_2\cdot t}\right]$$

$$z := 1.2 \qquad k := \frac{a_1\cdot a_2}{z} \qquad y_5(t) := \left[\frac{k\cdot z}{a_1\cdot a_2} + \frac{k\cdot(z-a_1)}{a_1\cdot(a_1-a_2)}\cdot e^{-a_1\cdot t} + \frac{k\cdot(z-a_2)}{a_2\cdot(a_2-a_1)}\cdot e^{-a_2\cdot t}\right]$$

$$z := 0.8 \qquad k := \frac{a_1\cdot a_2}{z} \qquad y_6(t) := \left[\frac{k\cdot z}{a_1\cdot a_2} + \frac{k\cdot(z-a_1)}{a_1\cdot(a_1-a_2)}\cdot e^{-a_1\cdot t} + \frac{k\cdot(z-a_2)}{a_2\cdot(a_2-a_1)}\cdot e^{-a_2\cdot t}\right]$$

$$z := 0.6 \qquad k := \frac{a_1\cdot a_2}{z} \qquad y_7(t) := \left[\frac{k\cdot z}{a_1\cdot a_2} + \frac{k\cdot(z-a_1)}{a_1\cdot(a_1-a_2)}\cdot e^{-a_1\cdot t} + \frac{k\cdot(z-a_2)}{a_2\cdot(a_2-a_1)}\cdot e^{-a_2\cdot t}\right]$$

$$z := -1.6 \qquad k := \frac{a_1\cdot a_2}{z} \qquad y_8(t) := \left[\frac{k\cdot z}{a_1\cdot a_2} + \frac{k\cdot(z-a_1)}{a_1\cdot(a_1-a_2)}\cdot e^{-a_1\cdot t} + \frac{k\cdot(z-a_2)}{a_2\cdot(a_2-a_1)}\cdot e^{-a_2\cdot t}\right]$$



Some Important Features:

1) The zero (z) is in the LHP if z is positive.

2) If the zero is closer to the origin than the poles, than it can cause overshoot and/or significant steady-state error.

Remember this one

3) The steady-state error will be 100% (no DC gain) if the zero is at the origin. The zero is at the origin cancels the pole of the DC (step) input. (The system has a differentiator.)

4) A zero in the RHP (non-minimum phase zero) can cause undershoot or a negative DC gain.