

Partial Fraction Expansion

**Ex.1**  $F(z) = \frac{1}{(z-1)\cdot(z+1)}$  Example 1 from Bodson, page 197

Divide by z first, because all the table entries have a z in the numerator, you can remultiply by z at the end.

$$\frac{F(z)}{z} = \frac{1}{z\cdot(z-1)\cdot(z+1)} = \frac{A}{z} + \frac{B}{(z-1)} + \frac{C}{z+1}$$

Multiply both sides by:  $z\cdot(z-1)\cdot(z+1)$

$$1 = A\cdot(z-1)\cdot(z+1) + B\cdot z\cdot(z+1) + C\cdot z\cdot(z-1)$$

Set  $z := 0$

$$1 = A\cdot(0-1)\cdot(0+1) + 0 + 0 \quad A := \frac{1}{-1} \quad A = -1$$

Set  $z := 1$

$$1 = 0 + B\cdot 1\cdot(1+1) + 0 \quad B := \frac{1}{2} \quad B = 0.5$$

Set  $z := -1$

$$1 = 0 + 0 + C\cdot(-1)\cdot(-1-1) \quad C := \frac{1}{2} \quad C = 0.5$$

$$\frac{F(z)}{z} = \frac{1}{z\cdot(z-1)\cdot(z+1)} = \frac{-1}{z} + \frac{1}{2}\cdot\frac{1}{(z-1)} + \frac{1}{2}\cdot\frac{1}{z+1}$$

Now multiply back through by z to get partial fractions that can actually be found in the table.

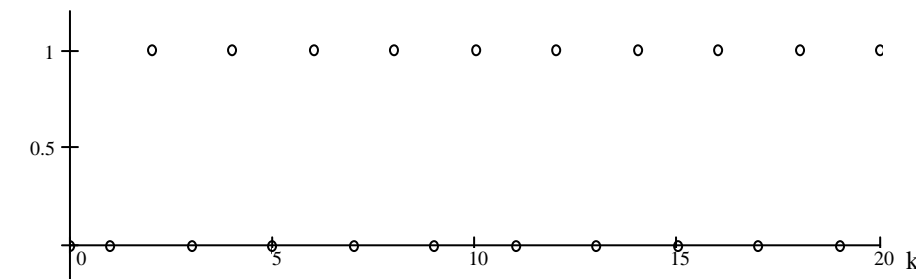
$$F(z) = \frac{1}{(z-1)\cdot(z+1)} = \frac{-1\cdot z}{z} + \frac{1}{2}\cdot\frac{z}{(z-1)} + \frac{1}{2}\cdot\frac{z}{z+1}$$

$$f(k) := \left[ -1\cdot\delta(k) + \frac{1}{2} + \frac{1}{2}\cdot(-1)^k \right] \cdot u(k)$$

By long division, as shown in section 6.3.2 in Bodson text.

$$(z-1)\cdot(z+1) = (z^2-1)$$

$$\begin{array}{r} z^{-2} + z^{-4} + z^{-6} + \dots \\ (z^2-1) \overline{) 1} \\ \underline{1 - z^2} \phantom{+ \dots} \\ z^{-2} \\ \underline{z^{-2} - z^{-4}} \phantom{+ \dots} \\ z^{-4} \\ \underline{z^{-4} - z^{-6}} \phantom{+ \dots} \\ z^{-6} \text{ etc} \end{array}$$



never ends

**Ex.2**  $F(z) = \frac{z}{(z - 0.9)^2 \cdot (z + 0.8)}$

$$\frac{F(z)}{z} = \frac{1}{(z - 0.9)^2 \cdot (z + 0.8)} = \frac{A}{z - 0.9} + \frac{0.9 \cdot B}{(z - 0.9)^2} + \frac{C}{z + 0.8}$$

Multiply both sides by:  $(z - 0.9)^2 \cdot (z + 0.8)$

$$1 = A \cdot (z - 0.9) \cdot (z + 0.8) + 0.9 \cdot B \cdot (z + 0.8) + C \cdot (z - 0.9)^2$$

Set  $z := 0.9$

$$1 = 0 + 0.9 \cdot B \cdot (0.9 + 0.8) + 0$$

$$B := \frac{1}{0.9 \cdot 1.7}$$

$$B = 0.654$$

Set  $z := -0.8$

$$1 = 0 + 0 + C \cdot (-0.8 - 0.9)^2$$

$$1 = C \cdot (-1.7)^2$$

$$C := \frac{1}{(-1.7)^2}$$

$$C = 0.346$$

Back to equation above

$$1 = A \cdot (z + 0.9) \cdot (z + 0.8) + 0.9 \cdot B \cdot (z + 0.8) + C \cdot (z + 0.9)^2$$

$$1 = A \cdot z^2 + 1.7 \cdot A \cdot z + .72 \cdot A + 0.9 \cdot B \cdot z + 0.72 \cdot B + C \cdot z^2 + 1.8 \cdot C \cdot z + .81 \cdot C$$

$$0 \cdot z^2 = A \cdot z^2 + 0 + C \cdot z^2$$

$$A := -C$$

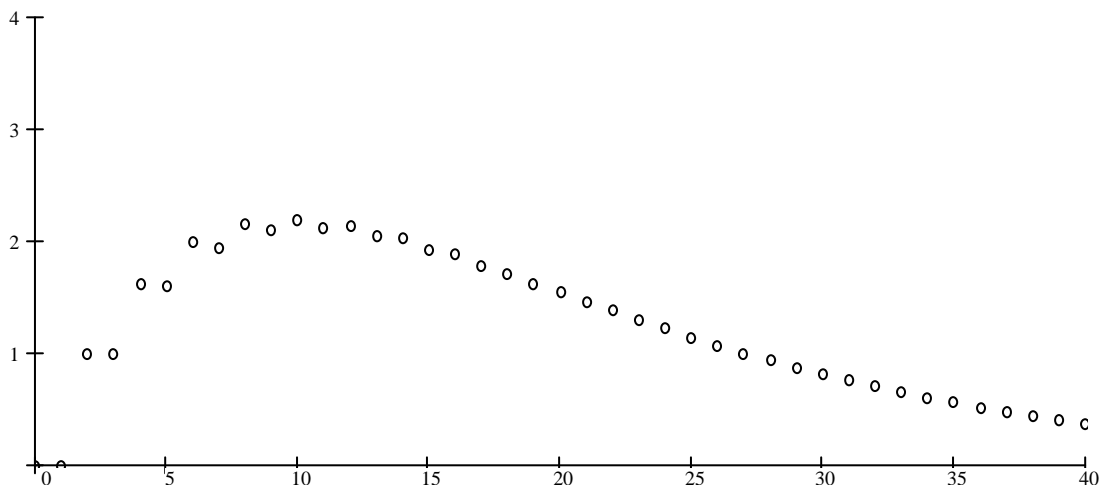
$$A = -0.346$$

no  $z^2$  term on the left

$$F(z) = \frac{z}{(z - 0.9)^2 \cdot (z + 0.8)} = \frac{-0.346 \cdot z}{z - 0.9} + \frac{0.654 \cdot 0.9 \cdot z}{(z - 0.9)^2} + \frac{0.346 \cdot z}{z + 0.8}$$

$$f(k) = -0.346 \cdot (0.9)^k + 0.654 \cdot k \cdot (0.9)^k + 0.346 \cdot (-0.8)^k$$

$$f(k) := [-0.346 \cdot (0.9)^k + 0.654 \cdot k \cdot (0.9)^k + 0.346 \cdot (-0.8)^k] \cdot u(k) \quad k := 0, 1 \dots 40$$



**Ex.3**  $F(z) = \frac{z}{z^2 - 2 \cdot z + 2}$

The complex coefficient way (not recommended)

$$\frac{F(z)}{z} = \frac{1}{(z - (1+j)) \cdot (z - (1-j))} = \frac{A}{(z - (1+j))} + \frac{B}{(z - (1-j))}$$

$$\frac{1}{(z - (1-j))} \Big|_{z = (1+j)} = A = \frac{1}{((1+j) - (1-j))} = -0.5j$$

$$\frac{1}{(z - (1+j))} \Big|_{z = 1-j} = B = \frac{1}{((1-j) - (1+j))} = 0.5j$$

$$\mathbf{p} := (1+j) \quad |\mathbf{p}| = \sqrt{2} \quad \angle \mathbf{p} = \theta_p = \frac{\pi}{4}$$

$$\mathbf{c} = -\frac{1}{2} \cdot j \quad |\mathbf{c}| = \frac{1}{2} \quad \angle \mathbf{c} = \theta_c = -\frac{\pi}{2}$$

Use this Table entry  $\frac{\mathbf{C} \cdot z}{(z - \mathbf{p})} + \frac{\bar{\mathbf{C}} \cdot z}{(z - \bar{\mathbf{p}})} \leftrightarrow 2 \cdot |\mathbf{C}| \cdot (|\mathbf{p}|)^k \cdot \cos(\theta_p \cdot k + \theta_c)$  Note: table shows B, where I've changed to C for clarity here

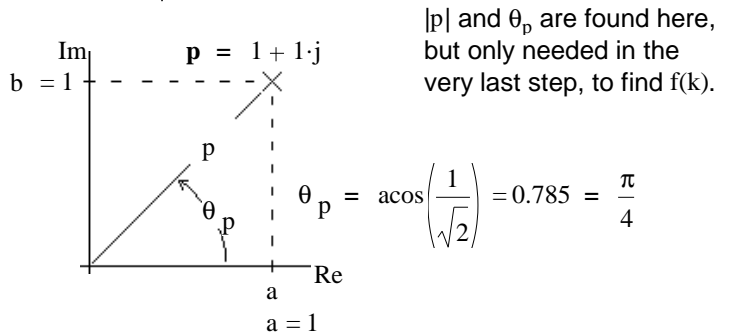
$$f(k) = 2 \cdot |\mathbf{c}| \cdot (|\mathbf{p}|)^k \cdot \cos(\theta_p \cdot k + \theta_c) = 2 \cdot \frac{1}{2} \cdot (\sqrt{2})^k \cdot \cos\left(\frac{\pi}{4} \cdot k - \frac{\pi}{2}\right) \cdot u(k) = (\sqrt{2})^k \cdot \sin\left(\frac{\pi}{4} \cdot k\right) \cdot u(k)$$

The easy way

$$p^k \cdot \cos(\theta_p \cdot k) \leftrightarrow \frac{z \cdot (z - a)}{z^2 - 2 \cdot a \cdot z + (a^2 + b^2)} = \frac{z \cdot (z - p \cdot \cos(\theta_p))}{z^2 - 2 \cdot p \cdot \cos(\theta_p) \cdot z + p^2}$$

$$p^k \cdot \sin(\theta_p \cdot k) \leftrightarrow \frac{z \cdot b}{z^2 - 2 \cdot a \cdot z + (a^2 + b^2)} = \frac{z \cdot (p \cdot \sin(\theta_p))}{z^2 - 2 \cdot p \cdot \cos(\theta_p) \cdot z + p^2}$$

Fit to our denominator:  $z^2 - 2 \cdot z + 2$   
 $a := 1 \quad b := \sqrt{2 - a^2} \quad b = 1 \quad p = \sqrt{2}$



Continue partial fraction expansion

$$\frac{F(z)}{z} = \frac{1}{z^2 - 2 \cdot z + 2} = \frac{A(z - 1)}{z^2 - 2 \cdot z + 2} + \frac{B(1)}{z^2 - 2 \cdot z + 2} \quad \text{Let: } z = 1 \quad B := 1$$

$$1 = A(z - 1) + B$$

$$0 \cdot z = A \cdot z \quad A := 0$$

$|\mathbf{p}|$  and  $\theta_p$  are needed here, to find  $f(k)$ .  $f(k) = A \cdot (p^k \cdot \cos(\theta_p \cdot k)) \cdot u(k) + B \cdot (p^k \cdot \sin(\theta_p \cdot k)) \cdot u(k)$

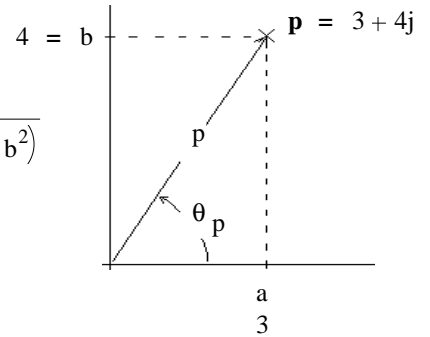
$$f(k) = (\sqrt{2})^k \cdot \sin\left(\frac{\pi}{4} \cdot k\right) \cdot u(k)$$

**Ex.4**

$$F(z) = \frac{2 \cdot z \cdot (3 \cdot z + 17)}{(z - 1) \cdot (z^2 - 6 \cdot z + 25)}$$

$$\begin{aligned} \frac{F(z)}{z} &= \frac{2 \cdot (3 \cdot z + 17)}{(z - 1) \cdot (z^2 - 6 \cdot z + 25)} = \frac{A}{z - 1} + \frac{B \cdot (z - a)}{z^2 - 2 \cdot a \cdot z + (a^2 + b^2)} + \frac{C \cdot b}{z^2 - 2 \cdot a \cdot z + (a^2 + b^2)} \\ &= \frac{A}{z - 1} + \frac{B(z - a)}{z^2 - 6 \cdot z + 25} + \frac{C \cdot b}{z^2 - 6 \cdot z + 25} \end{aligned}$$

$z^2 - 6 \cdot z + 25$   
 $z^2 - 2 \cdot a \cdot z + (a^2 + b^2)$   
 $a := 3 \qquad b := \sqrt{25 - a^2} \qquad b = 4$



$|p|$  and  $\theta_p$  are found here, but only needed in the very last step, to find  $f(k)$ .  
 $|p| = \sqrt{25} = 5$        $\theta_p = \text{asin}\left(\frac{4}{5}\right) = 0.927 = \text{acos}\left(\frac{3}{5}\right) = 0.927 = \text{atan}\left(\frac{4}{3}\right) = 0.927$  (in radians)  
 several ways to find  $\theta_p$  (in radians)

$$\frac{F(z)}{z} = \frac{2 \cdot (3 \cdot z + 17)}{(z - 1) \cdot (z^2 - 6 \cdot z + 25)} = \frac{A}{z - 1} + \frac{B(z - 3)}{z^2 - 6 \cdot z + 25} + \frac{C \cdot 4}{z^2 - 6 \cdot z + 25}$$

$$\left. \frac{2 \cdot (3 \cdot z + 17)}{(z^2 - 6 \cdot z + 25)} \right|_{z=1} = A = \frac{2 \cdot (3 \cdot 1 + 17)}{(1^2 - 6 \cdot 1 + 25)} = 2$$

$$2 \cdot (3 \cdot z + 17) = A \cdot (z^2 - 6 \cdot z + 25) + B \cdot (z - 3) \cdot (z - 1) + C \cdot 4 \cdot (z - 1)$$

$$6 \cdot z + 34 = 2 \cdot (z^2 - 6 \cdot z + 25) + B \cdot (z^2 - 4 \cdot z + 3) + C \cdot 4 \cdot (z - 1)$$

$$6 \cdot z + 34 = 2 \cdot z^2 - 12 \cdot z + 50 + B \cdot z^2 - 4 \cdot B \cdot z + 3 \cdot B + C \cdot 4 \cdot z - C \cdot 4$$

$$B := -2$$

$$6 \cdot z = -12 \cdot z + 4 \cdot 2 \cdot z + C \cdot 4 \cdot z$$

$$C = \frac{6 + 12 - 8}{4} = 2.5$$

OR  $\frac{34 - 50 + 6}{-4} = 2.5$

$$F(z) = \frac{2 \cdot z \cdot (3 \cdot z + 17)}{(z - 1) \cdot (z^2 - 6 \cdot z + 25)} = \frac{2 \cdot z}{z - 1} + \frac{-2 \cdot z \cdot (z - 3)}{z^2 - 6 \cdot z + 25} + \frac{2.5 \cdot 4}{z^2 - 6 \cdot z + 25}$$

$|p|$  and  $\theta_p$  are needed here, to find  $f(k)$ .

$$u(k) \qquad (|p|)^k \cdot \cos(\theta_p \cdot k) \qquad (|p|)^k \cdot \sin(\theta_p \cdot k)$$

$$f(k) = 2 \cdot u(k) - 2 \cdot 5^k \cdot \cos(0.927 \cdot k) + 2.5 \cdot 5^k \cdot \sin(0.927 \cdot k)$$

$$f(k) = \left( 2 \cdot u(k) - 2 \cdot 5^k \cdot \cos(0.927 \cdot k) + 2.5 \cdot 5^k \cdot \sin(0.927 \cdot k) \right) u(k)$$