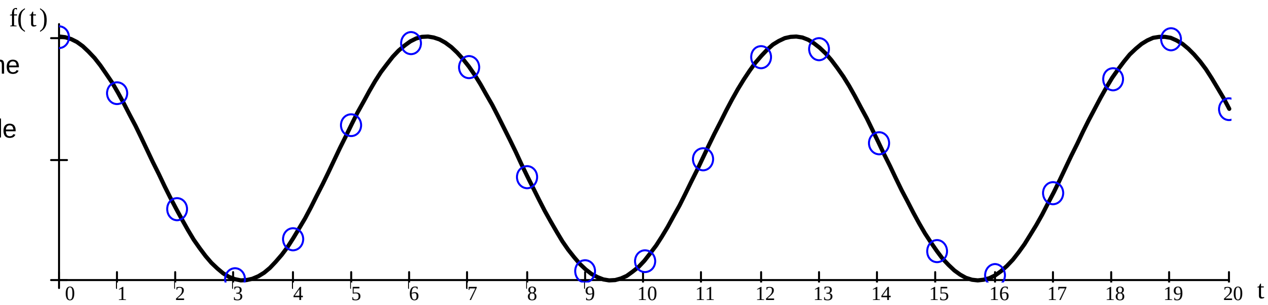
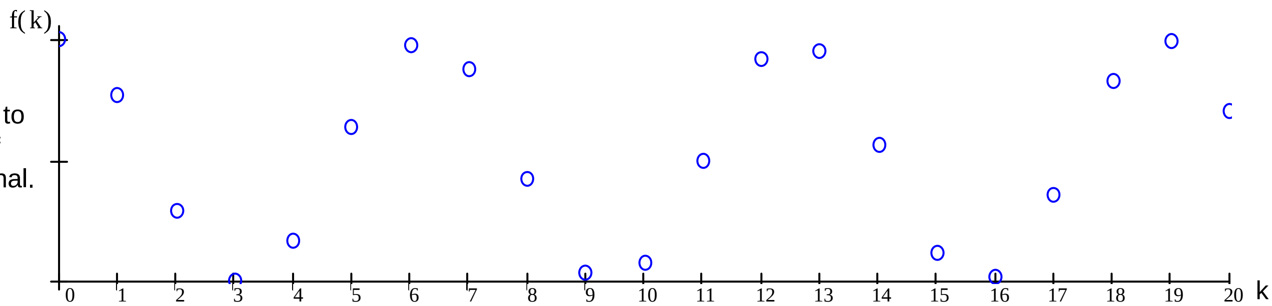
A Stolp
4/16/20,
4/18/23

Continuous-time (analog) signal showing sample times



Discrete-time (digital) signal corresponding to the samples of the analog signal.

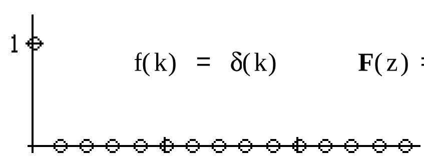


The z - transform

The z - transform will help us deal with discrete-time (digital) signals just like the Laplace transform helped us with continuous-time signals. So let's start making a table.

z - transform: $$\mathbf{F}(z) = \sum_{k=0}^{\infty} f(k)\cdot z^{-k}$$

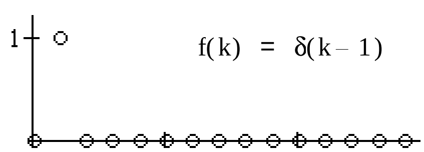
Impulse



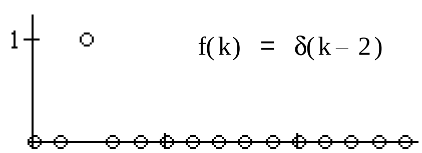
$f(k) = \delta(k)$ $\qquad \mathbf{F}(z) = \sum_{k=0}^{\infty} \delta(k)\cdot z^{-k} = 1 + 0 + 0 + 0 + \ldots$ $\qquad \mathbf{F}(z) = 1$ no pole

Just like Laplace: $f(t) = \delta(t)$ & $\mathbf{F}(s) = 1$

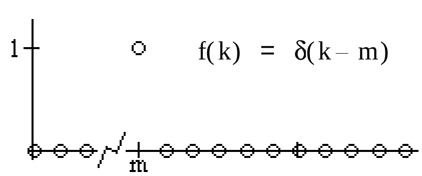
Delayed Impulses



$f(k) = \delta(k-1)$ $\qquad \mathbf{F}(z) = \sum_{k=0}^{\infty} \delta(k-1)\cdot z^{-k} = 0 + \frac{1}{z} + 0 + 0 + \ldots$ $\qquad \mathbf{F}(z) = \frac{1}{z}$



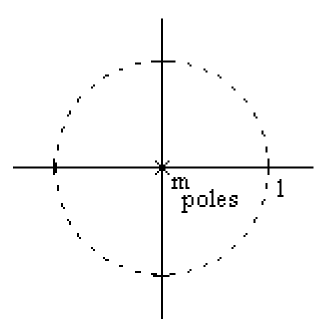
$f(k) = \delta(k-2)$ $\qquad \mathbf{F}(z) = \sum_{k=0}^{\infty} \delta(k-2)\cdot z^{-k} = 0 + 0 + \frac{1}{z^2} + 0 + \ldots$ $\qquad \mathbf{F}(z) = \frac{1}{z^2}$



$f(k) = \delta(k-m)$ $\qquad \mathbf{F}(z) = \sum_{k=0}^{\infty} \delta(k-m)\cdot z^{-k}$

$= 0 + \ldots 0 + \frac{1}{z^m} + 0 + 0 + \ldots = \frac{1}{z^m}$

$\mathbf{F}(z) = \frac{1}{z^m}$



Any finite-length signal can be made of delayed impulses, so all its poles are at the origin.

<u>Power Series</u> (Needed to solve the next few cases)

$$\text{SUM} = \sum_{k=0}^{n} \alpha^k = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \ldots \ \alpha^n$$

$$\text{SUM}\cdot(1-\alpha) = (1-\alpha)\left(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \ldots \ \alpha^n\right)$$

$$= 1\left(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \ldots \ \alpha^n\right) - \alpha\left(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \ldots \ \alpha^n\right)$$

$$= \begin{array}{l} 1\left(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \ldots \ \alpha^n\right) \\ -\left(\quad \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \ldots \ \alpha^n + \alpha^{n+1}\right) \\ \hline 1 \qquad\qquad\qquad\qquad\qquad\qquad -\alpha^{n+1} \end{array}$$

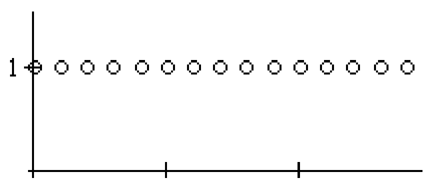
$$\text{SUM}\cdot(1-\alpha) = 1 - \alpha^{n+1}$$

$$\text{SUM} = \frac{1-\alpha^{n+1}}{(1-\alpha)} \qquad \text{If } n = \infty \quad \text{SUM} = \sum_{k=0}^{\infty} \alpha^{\infty} = \frac{1-\alpha^{\infty+1}}{(1-\alpha)} \overset{\text{if } (\alpha < 1)}{=} \frac{1}{(1-\alpha)}$$

in region of convergence ($\alpha < 1$)

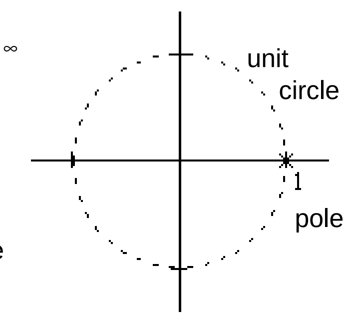
<u>Unit Step</u>

$$f(k) = u(k) \qquad \mathbf{F}(z) = \sum_{k=0}^{\infty} u(k)\cdot z^{-k} = 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \ldots \left(\frac{1}{z}\right)^{\infty}$$



$$= \frac{1}{\left(1-\frac{1}{z}\right)} = \mathbf{F}(z) = \frac{z}{(z-1)}$$

in region of convergence

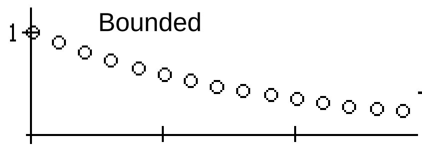


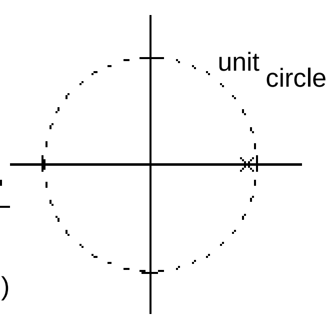
Like Laplace:
$f(t) = u(t)$ & $\mathbf{F}(s) = \frac{1}{s}$

<u>Geometric Progression</u>

$$f(k) = a^k\cdot u(k) \qquad \mathbf{F}(z) = \sum_{k=0}^{\infty} a^k\cdot z^{-k} = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \ldots \left(\frac{a}{z}\right)^{\infty} = \frac{1}{\left(1-\frac{a}{z}\right)} = \mathbf{F}(z) = \frac{z}{(z-a)}$$

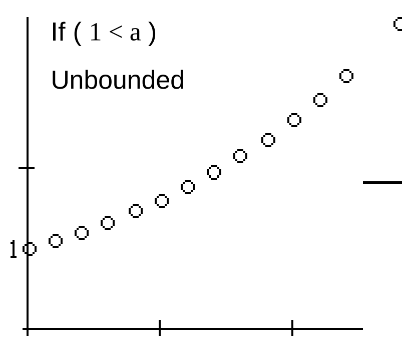
If ($a < 1$)

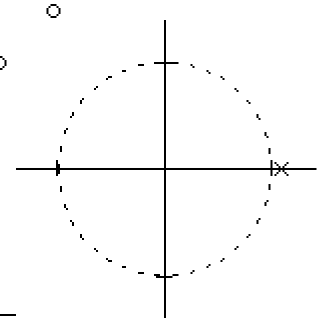




Unit step is a special case: ($a = 1$)

If ($1 < a$)



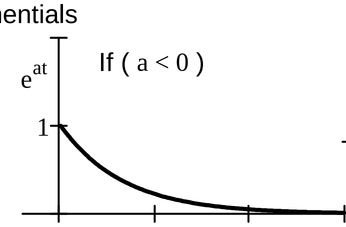


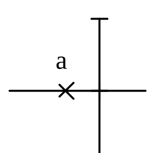
Like Laplace exponentials

$f(t) = e^{a\cdot t}\cdot u(t)$

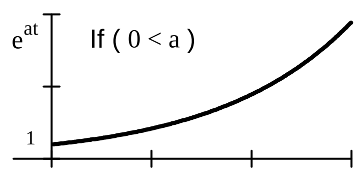
$\mathbf{F}(s) = \dfrac{1}{s-a}$

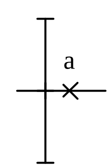
If ($a < 0$)





If ($0 < a$)

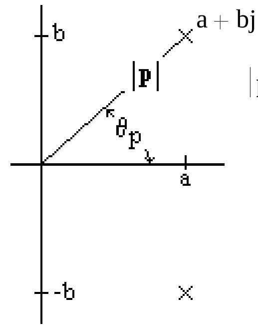




Sinusoidals

Complex Pole $\mathbf{p} = |\mathbf{p}|\cdot e^{j\cdot\theta_p} = a + bj$

Complex conjugate Pole $\bar{\mathbf{p}} = |\mathbf{p}|\cdot e^{-j\cdot\theta_p} = a - bj$



$|\mathbf{p}| = p$

Cosine

$$f(k) = \left[\mathbf{p}^k + \left(\bar{\mathbf{p}}\right)^k\right]\cdot u(k)$$

$$= \left[\left(|\mathbf{p}|\cdot e^{j\cdot\theta_p}\right)^k + \left(|\mathbf{p}|\cdot e^{-j\cdot\theta_p}\right)^k\right]\cdot u(k)$$

$$= \left(p^k\cdot e^{j\cdot\theta_p\cdot k} + p^k\cdot e^{-j\cdot\theta_p\cdot k}\right)\cdot u(k)$$

$$= p^k\cdot\left(e^{j\cdot\theta_p\cdot k} + e^{-j\cdot\theta_p\cdot k}\right)\cdot u(k)$$

$$= 2\cdot p^k\cdot\left(\frac{e^{j\cdot\theta_p\cdot k} + e^{-j\cdot\theta_p\cdot k}}{2}\right)\cdot u(k)$$

$$= 2\cdot p^k\cdot\cos\left(\theta_p\cdot k\right)\cdot u(k)$$

$$F(z) = \frac{z}{z-\mathbf{p}} + \frac{z}{z-\bar{\mathbf{p}}} \qquad \text{Linearity}$$

$$= \frac{z}{(z-a-bj)} + \frac{z}{(z-a+bj)}$$

$$= \frac{z\cdot(z-a+bj) + z\cdot(z-a-bj)}{(z-a-bj)\cdot(z-a+bj)}$$

$$= \frac{z\cdot(z-a+bj+z-a-bj)}{z^2 + (-a-bj)\cdot z + (-a+bj)\cdot z + (-a-bj)\cdot(-a+bj)}$$

$$= \frac{z\cdot(z-a+z-a)}{z^2 - 2\cdot a\cdot z + \left[(-a)^2 - bj^2\right]}$$

$$= \frac{2\cdot z\cdot(z-a)}{z^2 - 2\cdot a\cdot z + a^2 + b^2}$$

$$p^k\cdot\cos\left(e^{j\cdot\theta_p}\right)\cdot u(k) \quad \Longleftrightarrow \quad \frac{z\cdot(z-a)}{z^2 - 2\cdot a\cdot z + a^2 + b^2} = \frac{z\cdot(z-a)}{z^2 - 2\cdot a\cdot z + p^2}$$

$$\text{if} \quad |\mathbf{p}| = p = 1 \qquad \cos\left(e^{j\cdot\theta_p}\right)\cdot u(k) \quad \Longleftrightarrow \quad \frac{z\cdot(z-a)}{z^2 - 2\cdot a\cdot z + 1}$$

Sine

$$f(k) = \left[-j\cdot\mathbf{p}^k + j\cdot\left(\bar{\mathbf{p}}\right)^k\right]\cdot u(k)$$

$$= \left[e^{-j\cdot 90\cdot deg}\cdot\left(|\mathbf{p}|\cdot e^{j\cdot\theta_p}\right)^k + e^{j\cdot 90\cdot deg}\cdot\left(|\mathbf{p}|\cdot e^{-j\cdot\theta_p}\right)^k\right]\cdot u(k)$$

$$= \left[e^{-j\cdot 90\cdot deg}\cdot\left(p^k\cdot e^{j\cdot\theta_p\cdot k}\right) + e^{j\cdot 90\cdot deg}\cdot\left(p^k\cdot e^{-j\cdot\theta_p\cdot k}\right)\right]\cdot u(k)$$

$$= \left[p^k\cdot e^{j\cdot\left(\theta_p\cdot k - 90\cdot deg\right)} + p^k\cdot e^{-j\cdot\left(\theta_p\cdot k - 90\cdot deg\right)}\right]\cdot u(k)$$

$$= 2\cdot p^k\cdot\left[\frac{e^{j\cdot\left(\theta_p\cdot k - 90\cdot deg\right)} + e^{-j\cdot\left(\theta_p\cdot k - 90\cdot deg\right)}}{2}\right]\cdot u(k)$$

$$= 2\cdot p^k\cdot\cos\left(\theta_p\cdot k - 90\cdot deg\right)\cdot u(k)$$

$$= 2\cdot p^k\cdot\sin\left(\theta_p\cdot k\right)\cdot u(k)$$

$$F(z) = \frac{-j\cdot z}{z-\mathbf{p}} + \frac{j\cdot z}{z-\bar{\mathbf{p}}} \qquad \text{Linearity}$$

$$= \frac{-j\cdot z}{(z-a-bj)} + \frac{j\cdot z}{(z-a+bj)}$$

$$= \frac{-j\cdot z\cdot(z-a+bj) + j\cdot z\cdot(z-a-bj)}{(z-a-bj)\cdot(z-a+bj)}$$

$$= \frac{z\cdot\left((-j\cdot(z-a) - j\cdot bj) + (j\cdot(z-a) + j\cdot(-bj))\right)}{z^2 + (-a-bj)\cdot z + (-a+bj)\cdot z + (-a-bj)\cdot(-a+bj)}$$

$$= \frac{z\cdot(b+b)}{z^2 - 2\cdot a\cdot z + \left[(-a)^2 - bj^2\right]}$$

$$= \frac{2\cdot z\cdot b}{z^2 - 2\cdot a\cdot z + a^2 + b^2}$$

$$p^k\cdot\sin\left(e^{j\cdot\theta_p}\right)\cdot u(k) \quad \Longleftrightarrow \quad \frac{z\cdot b}{z^2 - 2\cdot a\cdot z + a^2 + b^2} = \frac{z\cdot b}{z^2 - 2\cdot a\cdot z + p^2}$$

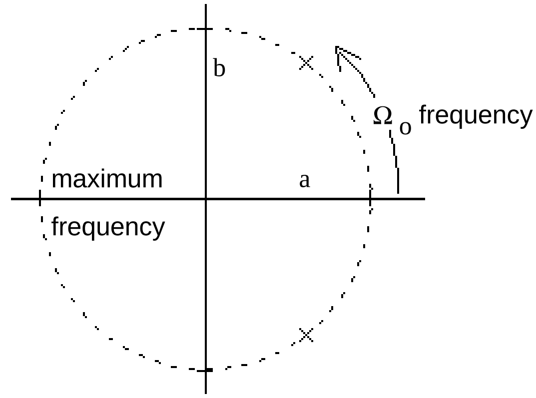
$$\text{if} \quad |\mathbf{p}| = p = 1 \qquad \sin\left(e^{j\cdot\theta_p}\right)\cdot u(k) \quad \Longleftrightarrow \quad \frac{z\cdot b}{z^2 - 2\cdot a\cdot z + 1}$$

<u>Sinusoidals</u>

$$f(k) = \cos(\Omega_o \cdot k)\cdot u(k) \qquad \mathbf{F}(z) = \frac{z(z - \cos(\Omega_o))}{z^2 - 2\cdot\cos(\Omega_o)\cdot z + 1} = \frac{z\cdot(z-a)}{z^2 - 2\cdot a\cdot z + 1}$$

AND

$$f(k) = \sin(\Omega_o \cdot k)\cdot u(k) \qquad \mathbf{F}(z) = \frac{z\cdot\sin(\Omega_o)}{z^2 - 2\cdot\cos(\Omega_o)\cdot z + 1} = \frac{z\cdot b}{z^2 - 2\cdot a\cdot z + 1}$$

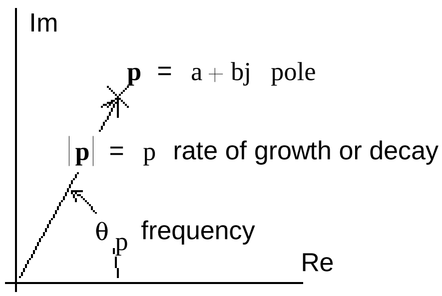


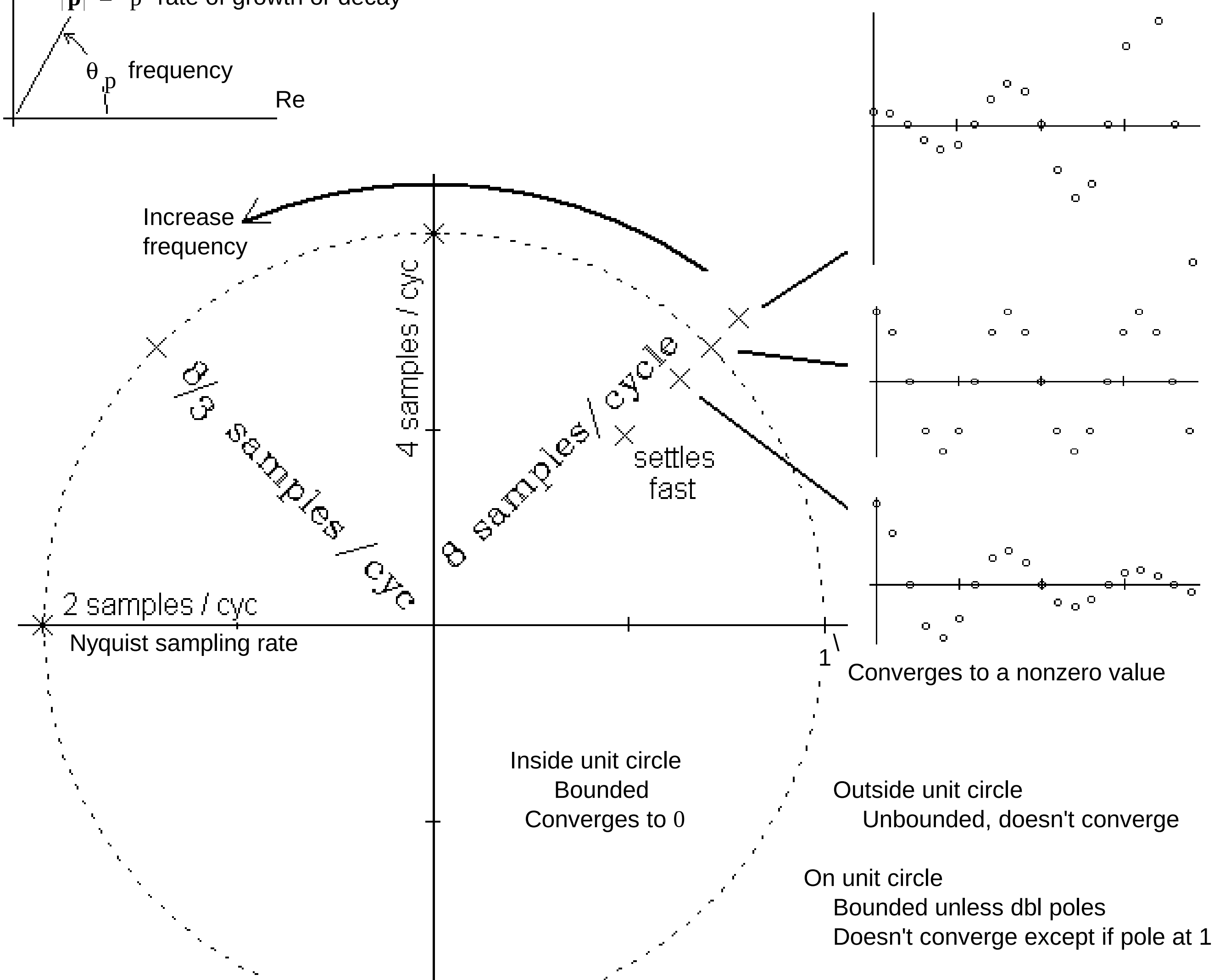
<u>Sinusoidals with growth or decay</u>

$$f(k) = p^k\cdot\cos(\theta_p\cdot k)\cdot u(k) \qquad \mathbf{F}(z) = \frac{z\cdot(z - p\cdot\cos(\theta_p))}{z^2 - 2\cdot p\cdot\cos(\theta_p)\cdot z + p^2} = \frac{z\cdot(z-a)}{z^2 - 2\cdot a\cdot z + p^2}$$

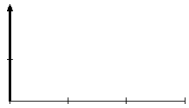
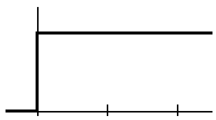
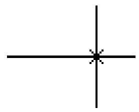
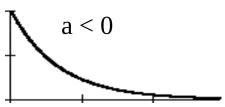
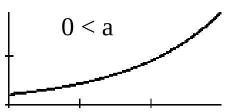
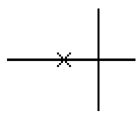
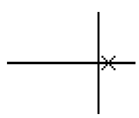
AND

$$f(k) = p^k\cdot\sin(\theta_p\cdot k)\cdot u(k) \qquad \mathbf{F}(z) = \frac{z\cdot(p\cdot\sin(\theta_p))}{z^2 - 2\cdot p\cdot\cos(\theta_p)\cdot z + p^2} = \frac{z\cdot b}{z^2 - 2\cdot a\cdot z + p^2}$$

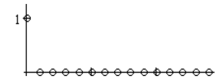
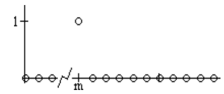
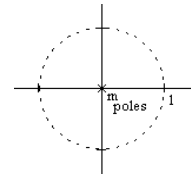
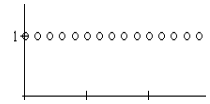
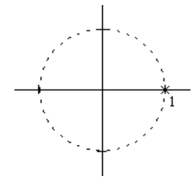
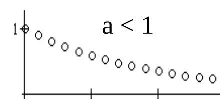
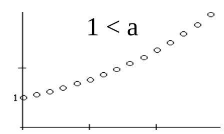
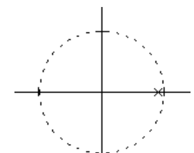
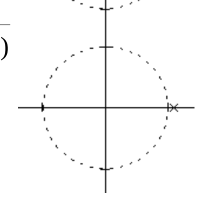




Laplace Transform (Unilateral)

	f(t)		F(s)	pole
impulse	$\delta(t)$		1	none
unit step	$u(t)$		$\frac{1}{s}$	
Exponential or Geometric Progression	$e^{a\cdot t}\cdot u(t)$	 	$\frac{1}{s-a}$	 

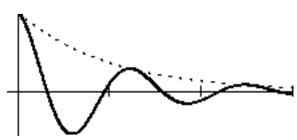
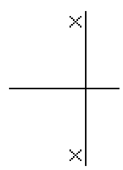
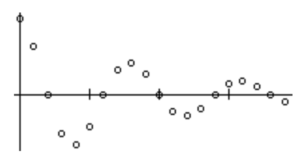
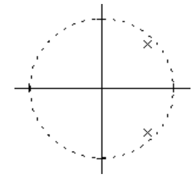
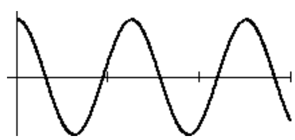
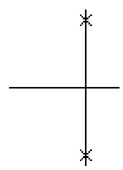
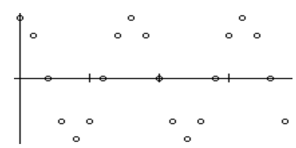
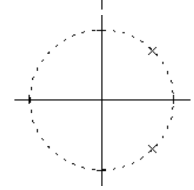
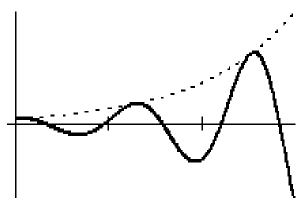
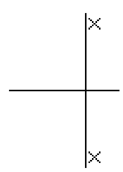
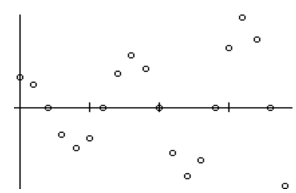
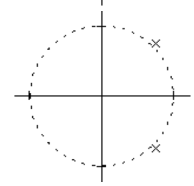
z - transforms

	f(k)		F(z)	pole
impulse	$\delta(k)$		1	none
delayed impulse	$\delta(k-m)$		$\frac{1}{z^m}$	
unit step	$u(k)$		$\frac{z}{z-1}$	
Exponential or Geometric Progression	$a^k\cdot u(k)$	 	$\frac{z}{(z-a)}$	 

Sinusoidals
Only cosine shown here

$e^{a\cdot t}\cdot\cos(b\cdot t)\cdot u(t) \quad \Longleftrightarrow \quad \frac{s-a}{(s-a)^2+b^2}$

$p^k\cdot\cos(\theta_p\cdot k)\cdot u(k) \quad \Longleftrightarrow \quad \frac{z\cdot(z-p\cdot\cos(\theta_p))}{z^2-2\cdot p\cdot\cos(\theta_p)\cdot z+p^2}$

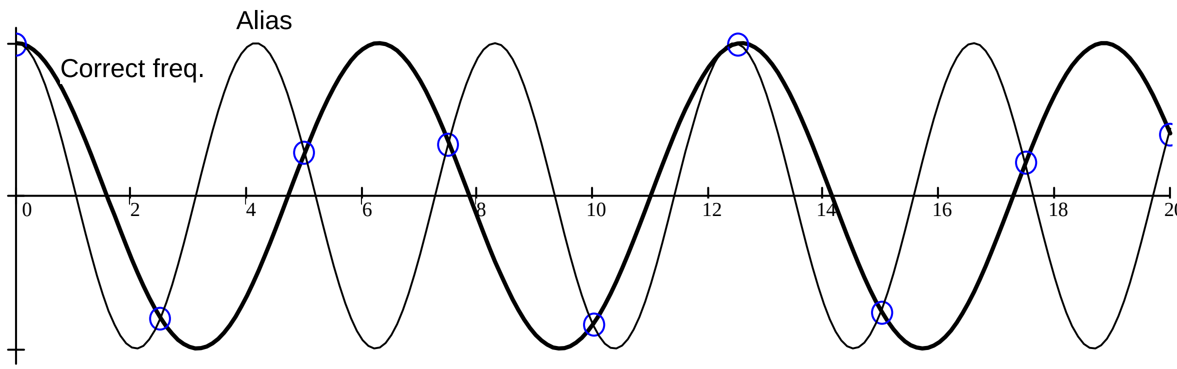
Laplace			z		
$a < 0$			$p < 1$		
$a = 0$			$p = 1$		
$0 < a$			$p > 1$		

Time constant $\tau = -\frac{1}{\ln(p)}$ Settling time $T_s = 4\cdot\tau$

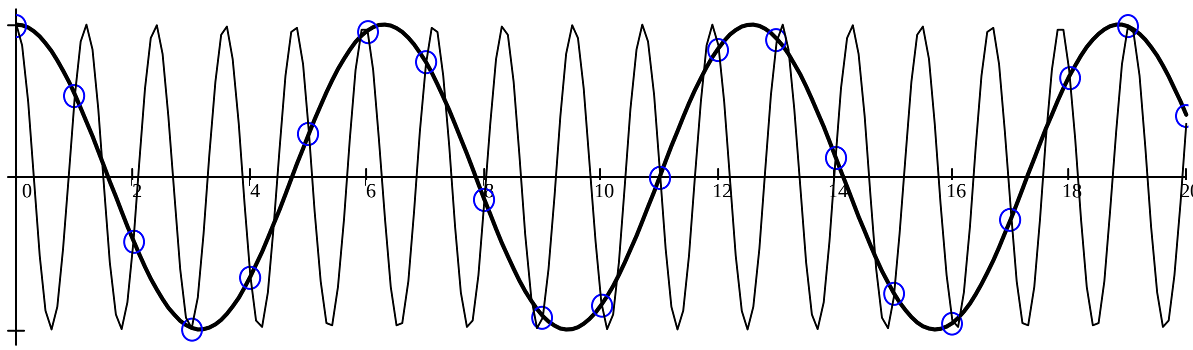
Damping factor $\zeta = \frac{-\ln(p)}{\sqrt{\ln(p)^2-\theta_p^2}}$

Aliasing

Close to the maximum frequency (2 samples per cycle)



Far from the maximum frequency.



z - transform Properties

All the following are multiplied by u(k) unless specified otherwise

Operation	f(k)	F(z)
Linearity	$c \cdot f(k) + d \cdot g(k)$	$c \cdot \mathbf{F}(z) + d \cdot \mathbf{G}(z)$
Right shift (Delay) $m \geq 0$	$f(k-m) \cdot u(k-m)$	$\frac{1}{z^m} \cdot \mathbf{F}(z) = z^{-m} \cdot \mathbf{F}(z)$
	$f(k-1)$	$z^{-1} \cdot \mathbf{F}(z) + f(-1)$
	$f(k-2)$	$z^{-2} \cdot \mathbf{F}(z) + z^{-1} \cdot f(-1) + f(-2)$
	$f(k-m)$	$\frac{1}{z^m} \cdot \mathbf{F}(z) + \frac{1}{z^m} \cdot \sum_{k=1}^{m} f(-k) \cdot z^k$
Left shift $m \geq 0$	$f(k+m)$	$z^m \cdot \mathbf{F}(z) - z^m \cdot \sum_{k=0}^{m-1} f(k) \cdot z^{-k}$
not common	$f(k+1)$	$z \cdot \mathbf{F}(z) - z \cdot f(0)$
	$f(k+2)$	$z^2 \cdot \mathbf{F}(z) - z^2 \cdot f(0) - z \cdot f(1)$
Initial value	$f(0)$	$\lim_{z \to \infty} \mathbf{F}(z)$
Final value	$f(\infty)$	$\lim_{z \to 1} (z-1) \cdot \mathbf{F}(z)$ (all poles of $(z-1)\mathbf{F}(z)$ inside unit circle)