CONTINUOUS JOINT DENSITY FUNCTION

__Defn__: Joint density function $f(x,y)$ of the continuous random variables X and Y

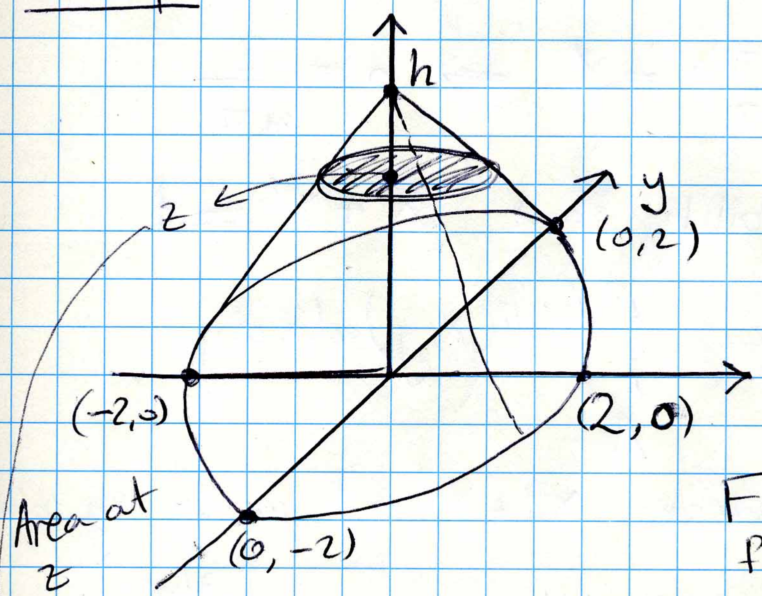
a) $f(x,y) \geq 0$ for all (x,y)

b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

Also for any region A in the xy plane

$$P[(X,Y) \in A] = \iint_A f(x,y)\,dx\,dy$$

__Example__:



$f(x,y)$ shaped like a cone

$$f(x,y) = \begin{cases} \dfrac{h\left(2-\sqrt{x^2+y^2}\right)}{2}, & \sqrt{x^2+y^2} \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

Find h that makes $f(x,y)$ a valid density.

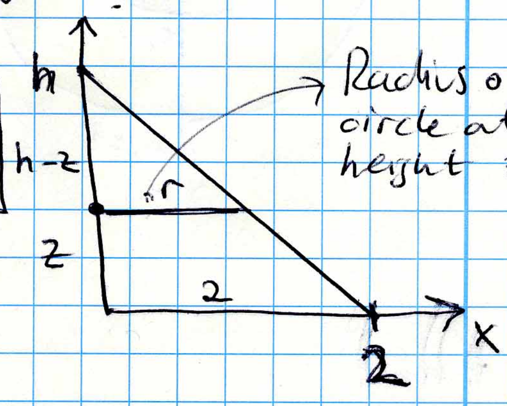
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \text{Volume under cone}$$

$$= \int_0^h (\text{Area of circle at height } z)\,dz$$

$$= \int_0^h \frac{4\pi(h-z)^2}{h^2}\,dz$$

$$\boxed{\frac{r}{2} = \frac{h-z}{h}}$$

similar triangles



Radius of circle at height z

$$\text{Vol} = \frac{4\pi}{h^2}\int_0^h h^2 - 2hz + z^2 \, dz$$

$$= \frac{4\pi}{h^2}\left(h^2 z \Big|_0^h - hz^2 \Big|_0^h + \frac{z^3}{3}\Big|_0^h \right)$$

$$= \frac{4\pi}{h^2}\left(h^3 - h^3 + \frac{h^3}{3} \right)$$

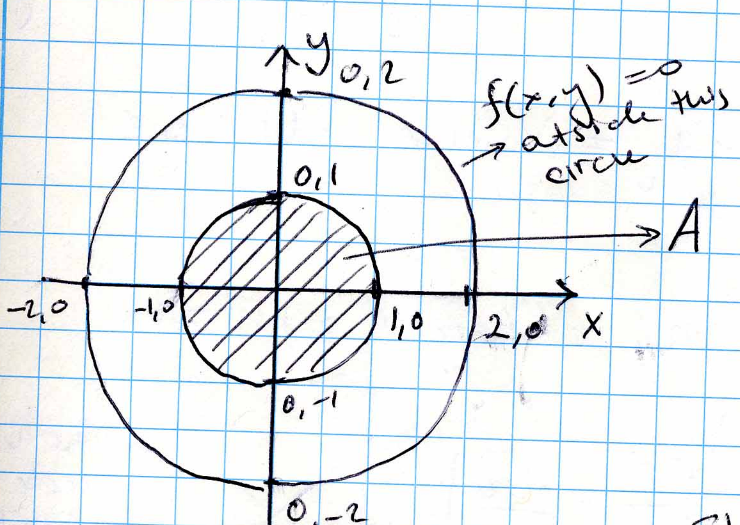
$$= \frac{4\pi h}{3}$$

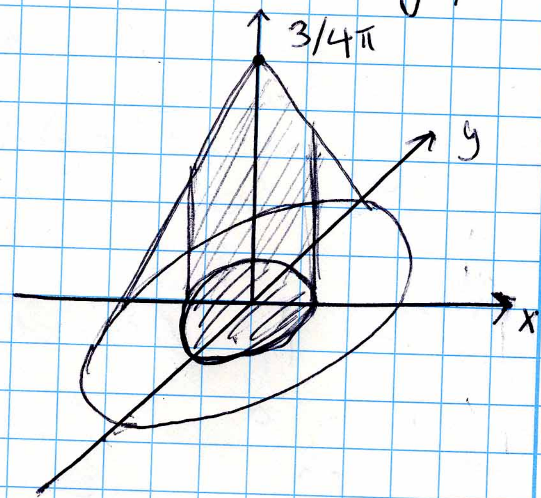
Since Vol = 1 $\quad \frac{4\pi h}{3} = 1 \Rightarrow h = \frac{3}{4\pi}$

* Find the probability that $X^2 + Y^2 \leq 1$

$$P(X^2+Y^2 \leq 1) = \iint_A f(x,y)\,dx\,dy$$

where A is the disk $X^2 + Y^2 \leq 1$ in the xy plane





Shaded volume is

$$P(X^2+Y^2 \leq 1) = \iint_A f(x,y)\,dx\,dy$$

Change to polar coordinates $r = \sqrt{x^2+y^2}$

$$\iint_A f(x,y)\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\,f(x,y)\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\,\frac{3}{8\pi}(2-r)\,dr\,d\theta$$

$$= \frac{3}{8\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} 2r - r^2\,dr\;d\theta$$

$$= \frac{3}{8\pi}\int_{0}^{2\pi}\left(r^2\Big|_0^1 - \frac{r^3}{3}\Big|_0^1 \right) d\theta$$

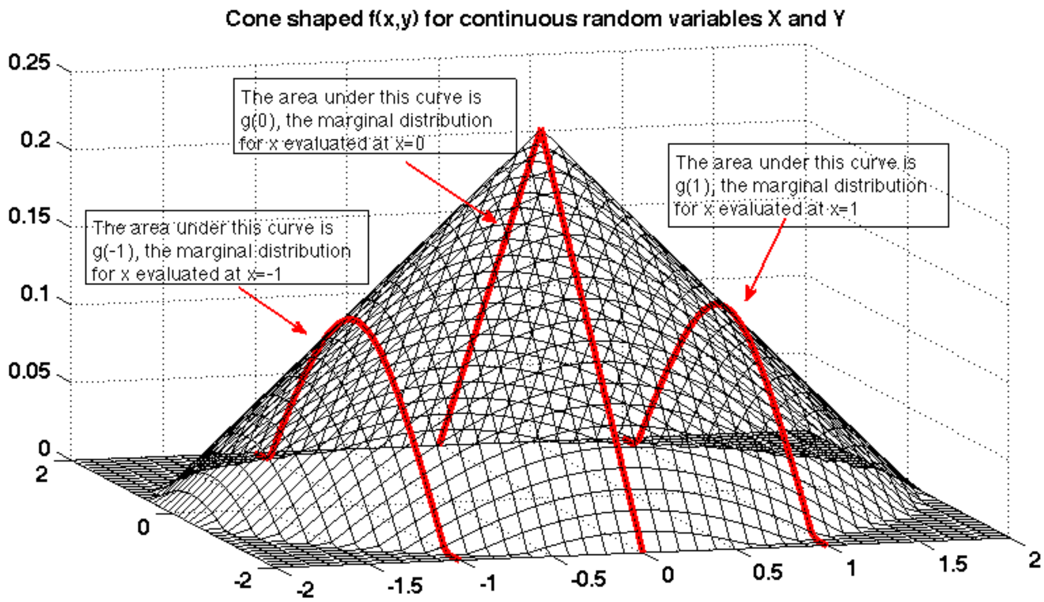
$$= \frac{3}{8\pi}\int_0^{2\pi}\left(1-\frac{1}{3}\right)d\theta = \frac{\cancel{3}}{\cancel{8}_4\pi}\times\frac{\cancel{2}}{\cancel{3}}\int_0^{2\pi} d\theta$$

$$= \frac{1}{4\pi}\times 2\pi = 1/2$$

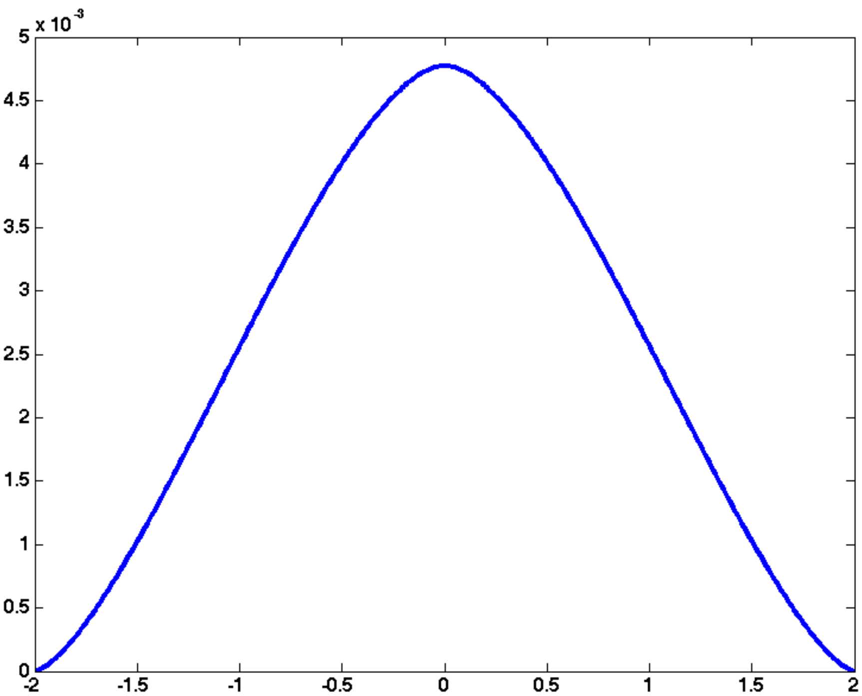
<u>Defn</u>: The marginal density functions of X alone and Y alone are

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy, \quad h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

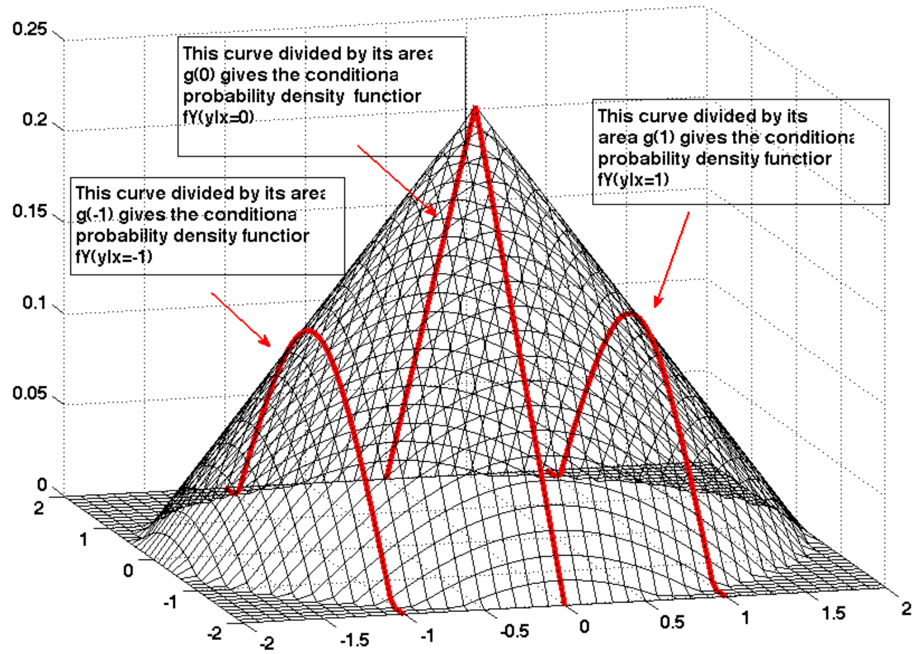
<u>Example</u>: see figures for marginal and conditional density examples with the cone shaped density function from previous example



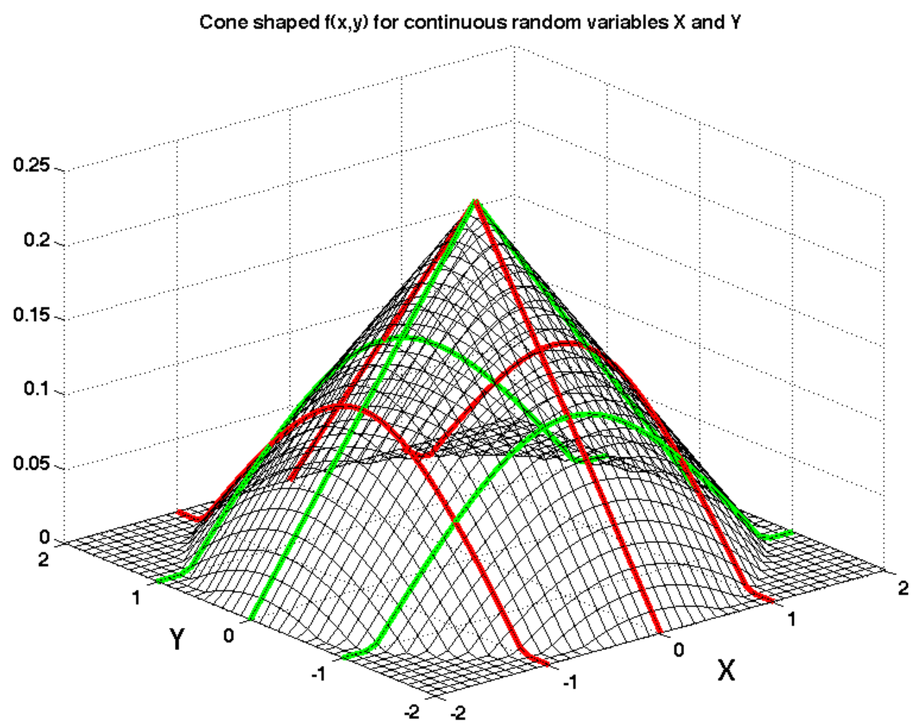
The areas under the red curves are the values for the marginal distribution g(x) evaluated at x=-1, x=0 and x=1



If we draw the red curves at each value of x, and for each compute the area underneath, we get the marginal distribution g(x) which we can then plot as a graph.

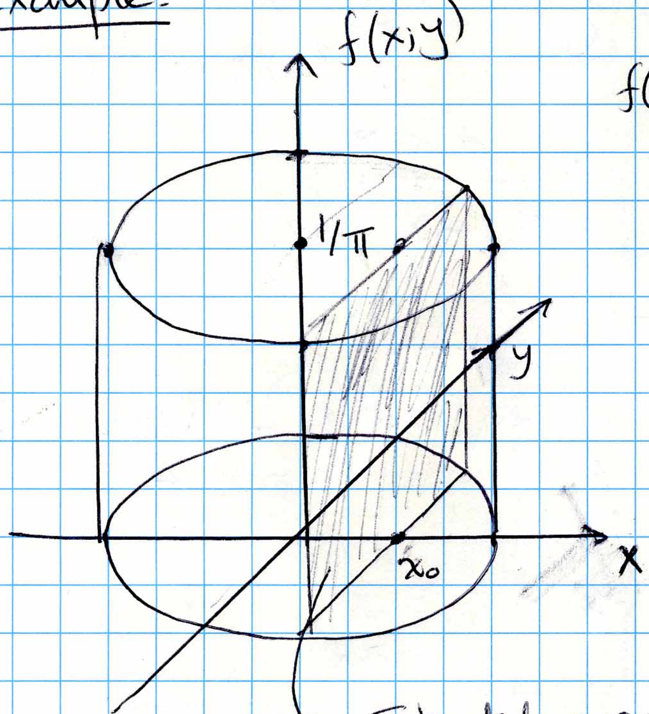


Once normalized by dividing with the appropriate value of g(x), the red curves are the conditional densities $f_Y(y|x=-1)$, $f_Y(y|x=0)$ and $f_Y(y|x=1)$.



The areas under the green curves are the values for the marginal distribution h(y) evaluated at y=-1, y=0 and y=1. Again if we normalize these curves by dividing with the appropriate value of h(y), the green curves become the conditional densities $f_X(x|y=-1)$, $f_X(x|y=0)$ and $f_X(x|y=1)$.

Example:



$$f(x,y) = \begin{cases} \dfrac{1}{\pi}, & x^2+y^2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

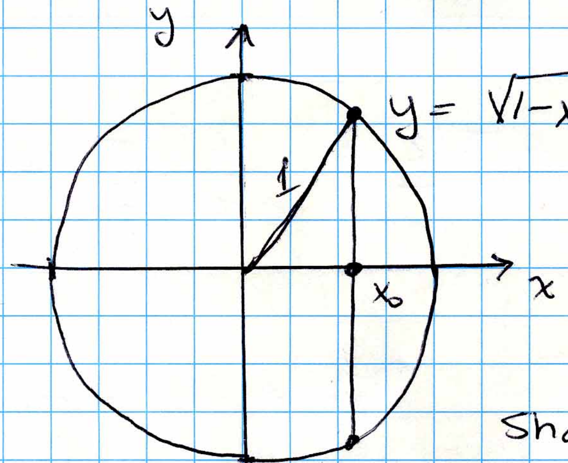
$$\text{Vol} = \frac{1}{\pi} \cdot \pi = 1 \checkmark$$

Compute the marginal densities.

Shaded area is $g(x_0)$.

Imagine collapsing the y axis.

$g(x)$ becomes the area under $f(x,y)$ for any given x.



$y = \sqrt{1-x_0^2}$

length of line from $(x_0, -\sqrt{1-x_0^2})$ to $(x_0, \sqrt{1-x_0^2})$ is

$$2\sqrt{1-x_0^2}$$

Shaded area is $\frac{1}{\pi} 2\sqrt{1-x_0^2}$

Therefore
$$g(x) = \begin{cases} \dfrac{2}{\pi}\sqrt{1-x^2}, & -1 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Also due to symmetry
$$h(y) = \begin{cases} \dfrac{2}{\pi}\sqrt{1-y^2}, & -1 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

You can check that $g(x)$ is a proper density function. We must have $\int_{-\infty}^{\infty} g(x)\,dx = 1$

$$\int_{-\infty}^{\infty} g(x)\,dx = \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\,dx$$

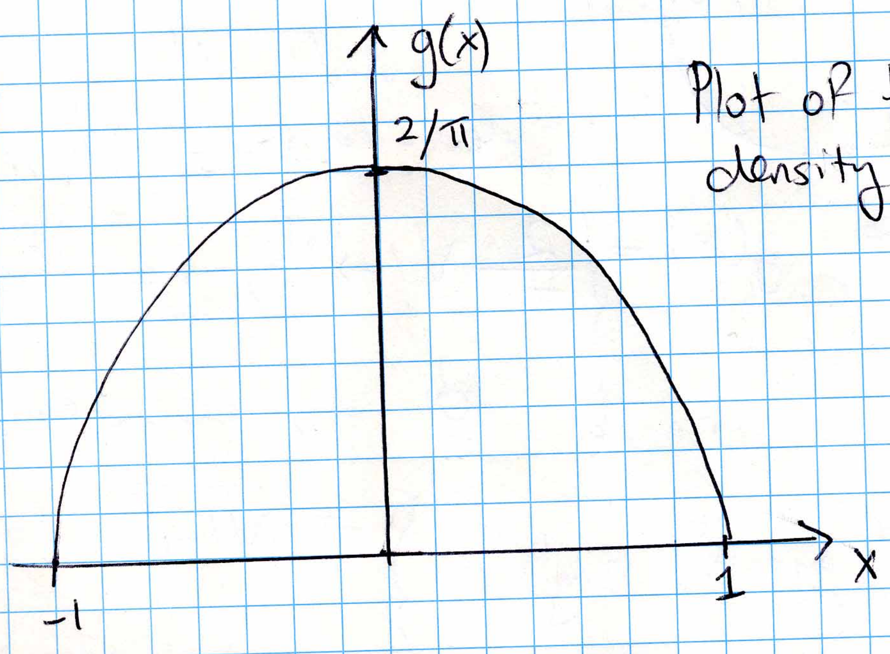
From an integral table $\int \sqrt{1-x^2}\,dx = \frac{x\sqrt{1-x^2}}{2} + \frac{\tan^{-1}\frac{x}{\sqrt{1-x^2}}}{2}$

So we have

$$\frac{2}{\pi}\left(\left.\frac{x\sqrt{1-x^2}}{2}\right|_{-1}^{1} + \left.\frac{\tan^{-1}\frac{x}{\sqrt{1-x^2}}}{2}\right|_{-1}^{1}\right)$$

$$= \frac{2}{\pi}\left(0 - 0 + \frac{\tan^{-1}\frac{1}{0} - \tan^{-1}-\frac{1}{0}}{2}\right)$$

$$= \frac{2}{\pi}\left(\frac{\pi/2 - (-\pi/2)}{2}\right) = \frac{2}{\pi}\times\frac{\pi}{2} = 1 \checkmark$$



Plot of the marginal density $g(x)$

Lets compute the conditional density $f_Y(y|x)$ for our example.

Defn: The conditional density functions are defined as

$$f_Y(y|x) = \frac{f(x,y)}{g(x)}, \quad g(x) > 0$$

$$f_X(x|y) = \frac{f(x,y)}{h(y)}, \quad h(y) > 0$$

$$f(x,y) = \begin{cases} \frac{1}{\pi}, & x^2+y^2 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

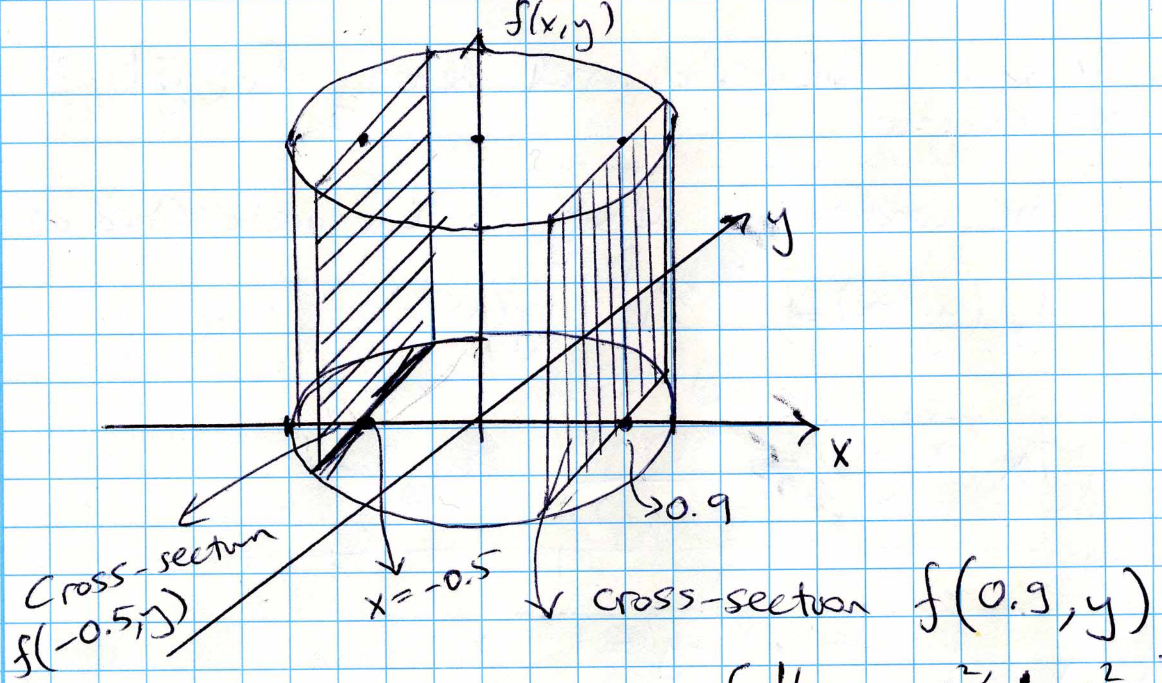
$$g(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2}, & -1 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

a) From the definition $f_Y(y|x)$ is defined only for those values of x for which $g(x) > 0$. For our example this is $-1 < x < 1$.

b) Once we fix a particular x, $f_Y(y|x)$ at that x is the cross-section of $f(x,y)$ at that x normalized by the area underneath the cross-section which is $g(x)$

With x fixed we can have the formula for the cross-section

$$\begin{matrix} \frac{1}{\pi} & \text{for} & y^2 \leq 1 - x^2 \\ 0 & & \text{otherwise.} \end{matrix}$$

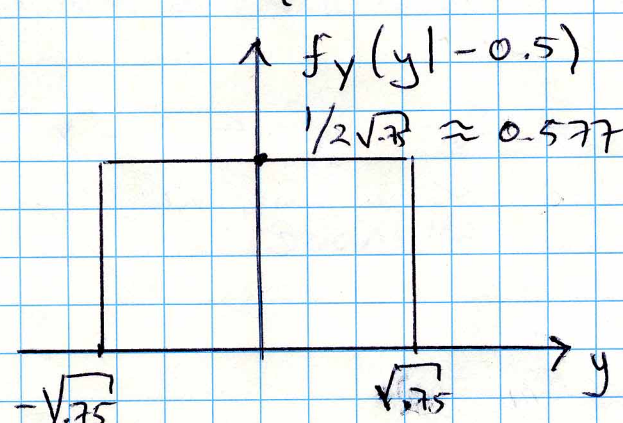


$$f_Y(y|x) = \frac{f(x,y)}{g(x)} = \frac{\begin{cases} 1/\pi, & y^2 \le 1-x^2 \\ 0, & \text{elsewhere} \end{cases}}{\begin{cases} \frac{2}{\pi}\sqrt{1-x^2}, & -1 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}}$$

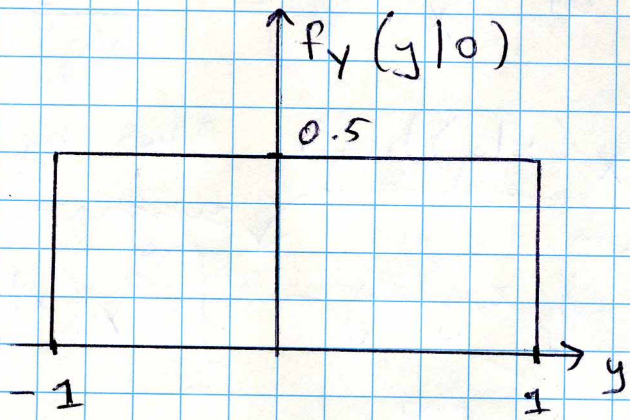
$$= \begin{cases} \dfrac{1}{2\sqrt{1-x^2}}, & y^2 \le 1-x^2 \text{ and } -1<x<1 \\ 0, & \text{elsewhere} \end{cases}$$

From this formula:

$$f_Y(y|-0.5) = \begin{cases} \dfrac{1}{2\sqrt{.75}}, & -\sqrt{.75} \le y \le \sqrt{.75} \\ 0, & \text{otherwise} \end{cases}$$



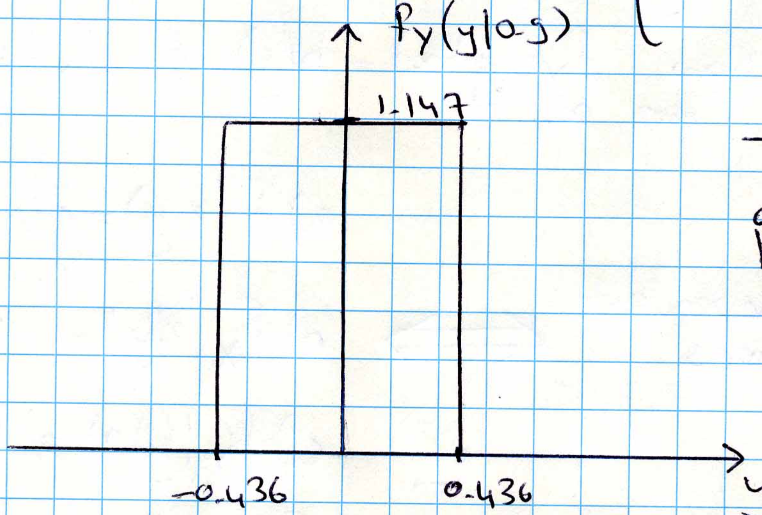
at $x=0$ $\quad f_Y(y|0) = \begin{cases} \frac{1}{2}, & -1 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$



Notice that this graph is wider and lower than the graph for $f_Y(y|-0.5)$

at $x = 0.9$

$$f_Y(y|0.9) = \begin{cases} 1.147, & \text{for } -0.436 \le y \le 0.436 \\ 0, & \text{otherwise} \end{cases}$$



This graph is different from both graphs we drew above

Since the conditional probability $f_Y(y|x)$ depends on x in this case we can say that the random variables X and Y are NOT independent.

The same is true with X and Y in the example before with the cone shaped $f(x,y)$.

Defn Independent random variables: If $f_X(x|y)$ does not depend on y, then $f_X(x|y) = g(x)$ and also $f(x,y) = g(x)h(y)$.

Proof: By definition of $f_X(x|y) = \dfrac{f(x,y)}{h(y)}$ we get

$$\underbrace{f(x,y) = f_X(x|y)h(y)}. \text{ Also by defn.}$$

substitute

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\infty}^{\infty} f_X(x|y)h(y)\,dy$$

$$= f_X(x|y) \underbrace{\int_{-\infty}^{\infty} h(y)\,dy}_{1} \quad \text{since } f_X(x|y) \text{ does NOT depend on } y.$$

$$= f_X(x|y)$$

Same proof to show $f_Y(y|x) = h(y)$ if $f_Y(y|x)$ does NOT depend on x.

X and Y independent random variables $\iff$
a) $f(x,y) = g(x)h(y)$
b) $f_X(x|y) = g(x)$
c) $f_Y(y|x) = h(y)$
for all (x,y)

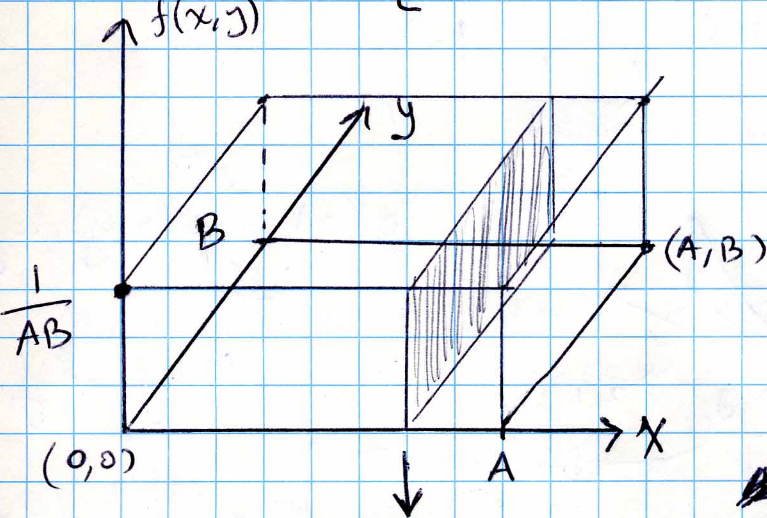
* Showing one of a, b or c holds for all (x,y) is enough to prove X, Y independent

* Showing one of a, b or c does NOT hold for some (x,y) is enough to show X, Y dependent

* If X, Y independent then all of a, b and c hold.

Example: Uniform distribution

$$f(x,y) = \begin{cases} \dfrac{1}{AB}, & 0 \le x \le A \text{ and } 0 \le y \le B \\ 0, & \text{elsewhere.} \end{cases}$$



NOTE: The defn. of $f_Y(y|x)$ requires $g(x)>0$ so $f_Y(y|x)$ exists only for $0 \le x \le A$ in this example. So when we say the cross-section is the same for all x, we really mean for all x in the range $0 \le x \le A$

The cross-section is the same for all x meaning $f_Y(y|x)$ does not depend on x which in turn means X, Y independent. Lets prove it.

Marginal densities: $g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^B \frac{1}{AB}\,dy = 1/A$

if $0 \le x \le A$

otherwise $g(x) = \int_{-\infty}^{\infty} 0\,dy = 0$

so $g(x) = \begin{cases} 1/A, & 0 \le x \le A \\ 0, & \text{otherwise} \end{cases}$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} \int_0^A \frac{1}{AB}\,dx = 1/B, & 0 \le y \le B \\ \int_{-\infty}^{\infty} 0\,dx = 0, & \text{elsewhere.} \end{cases}$$

$$g(x)h(y) = \begin{cases} 1/A \cdot 1/B & \text{if } 0 \le x \le A, \text{ and } 0 \le y \le B \\ 0, & \text{elsewhere.} \end{cases}$$

This is exactly $f(x,y)$.
INDEPENDENT.

Example: Let X and Y denote the position of an electron in the 2 dimensional Cartesian plane. Due to the uncertainty principle X and Y can't be measured exactly and are random variables. You are told that the measurement along the X-axis is independent from the measurement along the Y-axis. Furthermore, let X have a normal marginal density function with μ_X, σ_X and let Y have a normal marginal density function with μ_Y, σ_Y. What is the joint density function for X, Y?

Solution: The marginal density function for X is

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$$

The marginal density function for Y is

$$h(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}$$

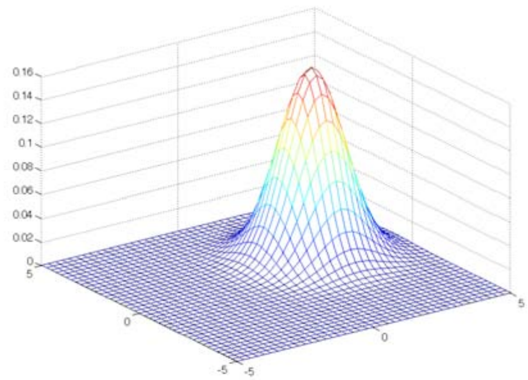
Using independence, we have $f(x, y) = g(x)h(y)$, so:

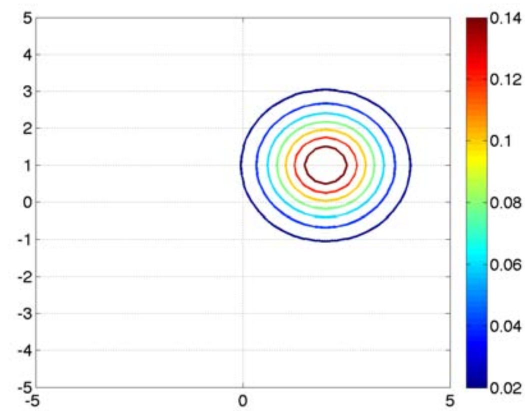
$$\begin{aligned} f(x,y) &= \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}} \\ &= \frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}} \end{aligned}$$

If we have $\sigma_X = \sigma = \sigma$, the joint density simplifies to

$$f(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x-\mu_X)^2+(y-\mu_Y)^2}{2\sigma^2}}$$

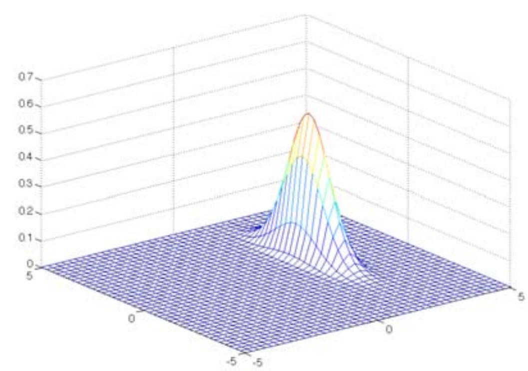
Here is what the joint density function $f(x, y)$ looks like for $\mu_X = 1$, $\mu_Y = 2$ and $\sigma_X = \sigma_Y = 1$.

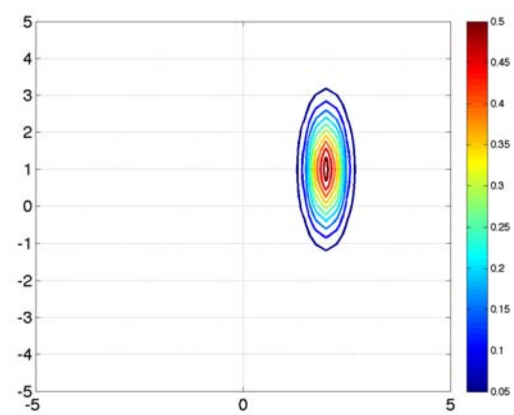




$f(x, y)$ Contours of constant probability.

Here is what the joint density function $f(x, y)$ looks like for $\mu_X = 1$, $\mu_Y = 2$ and $\sigma_X = 0.3$, $\sigma_Y = 1$.





$f(x, y)$ Contours of constant probability.

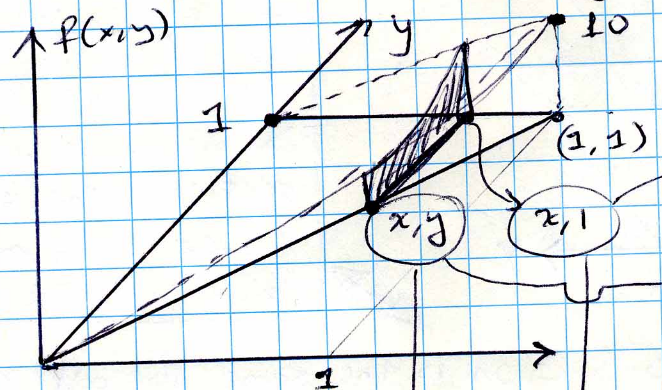
In this case there is more uncertainty in the Y position than the X position.

<u>Example</u> : (Example 3.19 textbook)

$$f(x,y) = \begin{cases} 10xy^2 & , 0<x<y<1 \\ 0 & , \text{elsewhere} \end{cases}$$

a) Find the marginal densities $g(x), h(y)$ and the conditional density $f_Y(y|x)$.

Lets first sketch $f(x,y)$



$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{x}^{1} 10xy^2\,dy$$

$$= \frac{10}{3}xy^3\Big|_x^1 = \frac{10}{3}x(1-x^3)$$

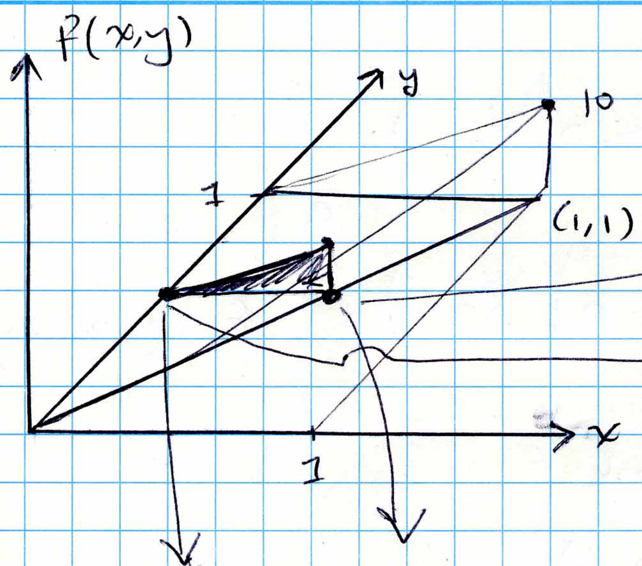
This is valid for $0<x<1$

Point A :
$x=y$ from slope of line connecting $(0,0)$ to $(1,1)$

Point B : $(x,1)$

Area of shaded cross-section is $g(x)$ for a particular x

$$g(x) = \begin{cases} \frac{10}{3}x(1-x^3) & , 0<x<1 \\ 0 & , \text{elsewhere} \end{cases}$$



$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^y 10xy^2\,dx$$

$$= 5x^2y^2 \Big|_0^y$$

$$= 5y^4$$

for $0<y<1$

$(x,y)=(0,y)$ $(x,y)=(y,y)$

Area of shaded cross-section is $h(y)$ for that particular y

$$h(y) = \begin{cases} 5y^4, & 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$$

$$f_Y(y|x) = \frac{f(x,y)}{g(x)} = \frac{10xy^2}{\frac{10}{3}x(1-x^3)} = \frac{3y^2}{1-x^3} \quad \text{for } 0<x<y<1$$

Notice that $f_Y(y|x)$ depends on x so X and Y are NOT independent. Equivalently, could show the same from $f(x,y) \neq g(x)h(y)$.

b) Find the probability that $Y > 1/2$ given $X=0.25$

$$P(Y>1/2 \mid X=0.25) = \int_{1/2}^{\infty} f_Y(y|x=0.25)\,dy$$

$$= \int_{1/2}^{1} \frac{3y^2}{1-0.25^3}\,dy = \frac{8}{9}$$

<u>Note:</u>

$$P(a<X<b \mid Y=y) = \int_a^b f_X(x|y)\,dx$$

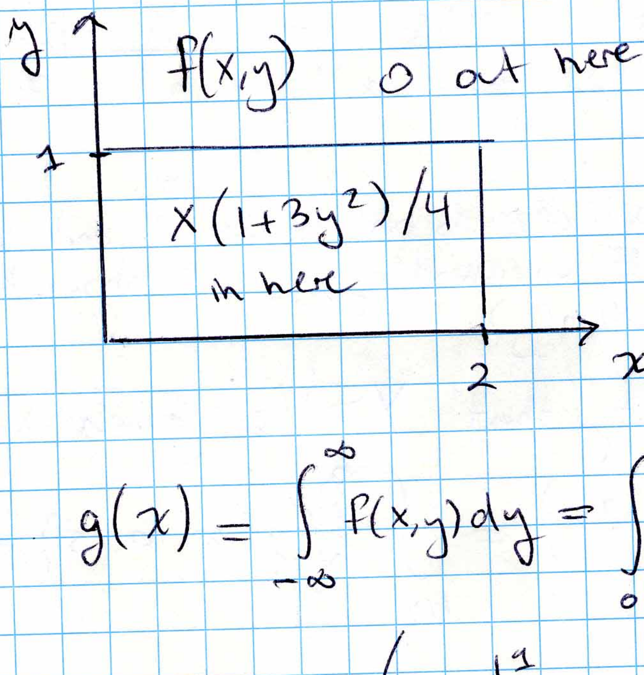
$$P(a<Y<b \mid X=x) = \int_a^b f_Y(y|x)\,dy$$

<u>Example</u> 3.20 from textbook

$$f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0<x<2,\ 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$$

a) Find the marginal densities $g(x)$, $h(y)$ and the conditional density $f_X(x|y)$?

Instead of sketching $f(x,y)$ lets sketch its footprint:



$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{x(1+3y^2)}{4}\,dy$$

$$= \frac{x}{4}\left(y\Big|_0^1 + \frac{y^3}{\cancel{4}}\Big|_0^1 \right) = \frac{x}{2} \quad \text{for } 0<x<2$$

outside $0<x<2$ $g(x)=0$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 \frac{x(1+3y^2)}{4}\,dx$$

$$= \frac{1+3y^2}{4}\int_0^2 x\,dx = \frac{1+3y^2}{4}\left(\frac{x^2}{2}\Big|_0^2\right)$$

$$= \frac{1+3y^2}{2}, \quad 0<y<1$$

Notice $f(x,y) = g(x)\,h(y)$ which means X and Y are independent.

$f_X(x|y) = g(x)$ since X and Y are independent

b) Compute the prob. that X is between $1/4$ and $1/2$ given that $Y = 1/3$.

$$P\left(\frac{1}{4} < X < \frac{1}{2} \,\Big|\, Y = 1/3\right) = \int_{1/4}^{1/2} f_X(x|y=1/3)\,dx$$

$$= \int_{1/4}^{1/2} \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_{1/4}^{1/2} = \frac{1}{\cancel{4}\ 16} - \frac{1}{\cancel{16}\ 64} = \frac{3}{64}$$